§5.4 Noether's theorem for fields

The Lagrangian density that we found for the string does not involve u explicitly, which only enters through its derivatives. This is a situation analogous to that of having an ignorable coordinate in the case of point particles. So we should expect that there is a symmetry associated to this fact, generated by the infinitesimal transformation $u \to u' = u + \epsilon$, and associated to this symmetry some conserved quantity, by some analogue for fields of Noether's theorem. This analogue does exist, as we now describe.

It is illuminating to do this more generally, for d spatial dimensions, and arbitrary symmetries. So let us introduce coordinates $x_0, \ldots, x_d$. The case $d = 1$ would have $x_0 = t$, $x_1 = x$. Our field $u(x_0, \ldots, x_d)$ is a map from $\mathbb{R}^{d+1} \to \mathbb{R}$. For convenience, we introduce the notation

$$u_i \coloneqq \frac{\partial u}{\partial x_i}\,.$$

Definition 5.4.1. A *symmetry* (in the context of field theory) is a transformation

$$u \to u' = u + \epsilon a(u)$$

such that

$$\delta\mathcal{L} = O(\epsilon^2)$$

without having to use the equations of motion.

Remark 5.4.2. We could include a total derivative, or more precisely, a divergence, on the right hand side of the variation $\delta\mathcal{L}$, as we did in the case of the point particle, but we ignore this possibility for simplicity.

Definition 5.4.3. We define the *generalised momentum vector*

$$\mathbf{\Pi} \coloneqq \left(\frac{\partial\mathcal{L}}{\partial u_0}, \ldots, \frac{\partial\mathcal{L}}{\partial u_d}\right)\,.$$

Definition 5.4.4. Given a transformation generated by a, we define the *Noether current* associated to the transformation by

$$\mathbf{J} \coloneqq a\mathbf{\Pi}\,,$$

or in components

$$J_i \coloneqq a\frac{\partial\mathcal{L}}{\partial u_i}\,.$$

Theorem 5.4.5 (Noether's theorem for fields). *If* $\mathbf{J}$ *is the Noether current associated to a symmetry, then*

$$\vec{\nabla}\cdot\mathbf{J} \coloneqq \sum_{i=0}^{d}\frac{\partial J_i}{\partial x_i} = 0\,. \tag{5.4.1}$$

Proof. We can proceed analogously as to what we did when proving Noether's theorem for discrete systems. Under a generic transformation we have

$$\delta\mathcal{L} = \epsilon a \frac{\partial \mathcal{L}}{\partial u} + \epsilon \sum_{i=0}^{d} \frac{\partial a}{\partial x_i} \frac{\partial \mathcal{L}}{\partial u_i} + O(\epsilon^2)\,.$$

Using the Euler-Lagrange equations, this becomes

$$\delta\mathcal{L} = \epsilon \sum_{i=0}^{d} \frac{\partial}{\partial x_i} \left(a \frac{\partial \mathcal{L}}{\partial u_i} \right) + O(\epsilon^2)$$

which equating with the explicit action of the symmetry on $\mathcal{L}$ leads to

$$\sum_{i=0}^{d} \frac{\partial}{\partial x_i} \left(a \frac{\partial \mathcal{L}}{\partial u_i} \right) = 0\,.$$

□

Definition 5.4.6. Given a Noether current $\mathbf{J}$ associated to a transformation, we define the (Noether) *charge density*

$$\mathcal{Q} := J_0\,.$$

Furthermore, in the $d = 1$ case (one spatial dimension)[14] we define the *charge contained in an interval* (a, b) to be

$$Q_{(a,b)} := \int_a^b \mathcal{Q}\, dx = \int_a^b J_0\, dx\,.$$

Proposition 5.4.7. *Assume* $d = 1$*. Then*

$$\frac{dQ_{(a,b)}}{dt} = J_1(a) - J_1(b)\,.$$

Proof. Taking the derivative inside the integral, we have

$$\begin{aligned} \frac{dQ_{(a,b)}}{dt} &= \frac{d}{dt} \int_a^b J_0\, dx \\ &= \int_a^b \frac{\partial J_0}{\partial t} dx\,. \end{aligned}$$

Now, in our $d = 1$ case the conservation equation (5.4.1) becomes

$$\frac{\partial J_0}{\partial t} + \frac{\partial J_1}{\partial x} = 0$$

[14]The $d > 1$ case can be treated similarly, with the total charge in some region being the integral of the function $\mathcal{Q}$ over that region.

so replacing $\frac{\partial J_0}{\partial t}$ by $-\frac{\partial J_1}{\partial x}$ inside the integral above we have

$$\frac{dQ_{(a,b)}}{dt} = -\int_a^b \frac{\partial J_1}{\partial x} dx = J_1(a) - J_1(b)\,.$$

□

Remark 5.4.8. The way to interpret proposition 5.4.7 is that it is telling us that the charge within some region changes only due to charge leaving or entering through the boundaries of the region. The current J_1 measures how much charge is leaving or entering by unit time on a given boundary component.

Definition 5.4.9. Given a Noether current $\mathbf{J}$ associated to a transformation, we define the *Noether charge* to be the total charge over all space. In the case of one spatial dimension ($d = 1$) this is

$$Q := Q_{(-\infty,\infty)} = \int_{-\infty}^{\infty} J_0 \, dx\,.$$

Corollary 5.4.10. *Assume that* $d = 1$, *and* $\lim_{x\to\pm\infty} J_1 = 0$. *Then*

$$\frac{dQ}{dt} = 0$$

for the Noether charge associated to a symmetry.

Proof. This follows immediately from proposition 5.4.7, since we assume $J_1(\pm\infty) = 0$. □

Example 5.4.11. *Let us apply all this abstract discussion to our guiding example, the one-dimensional string, and the symmetry arising from* u *being ignorable, namely* $u \to u + \epsilon$. *In this case we have* $a = 1$, *so the Noether current is simply given by*

$$\mathbf{J} = \mathbf{\Pi} = \left(\frac{\partial \mathcal{L}}{\partial u_t}, \frac{\partial \mathcal{L}}{\partial u_x}\right) = (\rho u_t, -\tau u_x)\,.$$

From here, we conclude that the Noether charge

$$Q = \int dx\, \mathbf{J}_0 = \rho \int dx\, u_t$$

is conserved in time, assuming that $J_1 = -\tau u_x$ *vanishes at infinity (in this case, since* τ *is a non-zero constant, this is equivalent to* u_x *vanishing at infinity). Indeed*

$$\frac{dQ}{dt} = \rho \int dx\, u_{tt} = \tau \int dx\, u_{xx} = \tau \big[u_x\big]_{-\infty}^{+\infty} = 0\,,$$

where in the middle step we have used the wave equation $\rho u_{tt} = \tau u_{xx}$ *for the string.*