§5.5 The Energy-Momentum Tensor

In addition to the conservation laws for transformations of the field itself, we also expect conservation laws associated to transformations of x and t. This is analogous to the fact that for systems with discrete degrees of freedom, we could construct an energy that satisfied

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}$$

Since t does not appear explicitly in the Lagrangian density for the string, we would expect energy to be conserved for oscillations of the string too. And indeed, it will prove quite easy to show that the total energy of the string is conserved. But the situation for the string is more interesting than that for the point particle. The string's energy is distributed along its length; some places may have no energy, whilst other parts of the string may be very energetic. As a wave packet travels, regions that had no energy may energise for some time, and then come back to having no energy. So we should not expect to have that the energy *density* at any given point is conserved. Additionally, in the case of fields the t and x directions are treated on equal footing, so there should be some generalised notion that treat the x variable the same as the t variable.

Definition 5.5.1. The *energy-momentum tensor* is

$$T_{ij} := \frac{\partial \mathcal{L}}{\partial u_j}\frac{\partial u}{\partial x_i} - \delta_{ij}\mathcal{L}\,. \tag{5.5.1}$$

Definition 5.5.2. The *energy density* $\mathcal{E}$ is defined to be equal to T_{00}.

Note 5.5.3

As for the case of the point particle, you can convince yourself that this definition of the energy density agrees with the ordinary one whenever the Lagrangian density is of the form $\mathcal{L} = \frac{1}{2}\rho u_t^2 - \frac{1}{2}\tau u_x^2 - \mathcal{V}(u)$; that is, a kinetic energy density minus a potential energy contribution (which in this case contains a possible contribution from the string tension, plus an additional term $\mathcal{V}(u)$ containing arbitrary extra contributions to the potential energy). See for instance example 5.5.6 below. In cases where the Lagrangian density is not of this form we can still define the energy-momentum tensor, and we simply *define* the energy density to be the T_{00} component.

Theorem 5.5.4. *The conservation laws for the energy-momentum tensor are:*

$$\sum_{j=0}^{d} \frac{\partial T_{ij}}{\partial x_j} = 0\,. \tag{5.5.2}$$

Proof. Consider the variation of the Lagrangian density $\mathcal{L}(u, u_0, \ldots, u_d)$ as we move in the x_i direction.[15] By the Chain Rule, this is given by

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial \mathcal{L}}{\partial u}\frac{\partial u}{\partial x_i} + \sum_{j=0}^{d} \frac{\partial \mathcal{L}}{\partial u_j}\frac{\partial^2 u}{\partial x_i \partial x_j}$$

Using the Euler-Lagrange equations for the field, we can rewrite this as

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_i} &= \left(\sum_{j=0}^{d} \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial u_j}\right)\right)\frac{\partial u}{\partial x_i} + \sum_{j=0}^{d} \frac{\partial \mathcal{L}}{\partial u_j}\frac{\partial^2 u}{\partial x_i \partial x_j} \\ &= \sum_{j=0}^{d} \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial u_j}\frac{\partial u}{\partial x_i}\right)\end{aligned}$$

or equivalently

$$\sum_{j=0}^{d} \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial u_j}\frac{\partial u}{\partial x_i} - \delta_{ij}\mathcal{L}\right) = 0\,.$$

□

Remark 5.5.5. Note that we have $d+1$ conservation equations for the energy-momentum tensor, one for each choice of "i".

Example 5.5.6. *This may look a little complicated, but it is not hard to evaluate in practice. For instance, for our string we have*

$$T_{tt} = u_t \frac{\partial \mathcal{L}}{\partial u_t} - \mathcal{L} = \frac{\rho}{2}(u_t)^2 + \frac{\tau}{2}(u_x)^2$$

which is indeed the energy density for the string. The rest of the components can be computed similarly, with the result

$$T = \begin{pmatrix} \frac{\rho}{2}(u_t)^2 + \frac{\tau}{2}(u_x)^2 & -\tau u_t u_x \\ \rho u_t u_x & -\frac{\rho}{2}(u_t)^2 - \frac{\tau}{2}(u_x)^2 \end{pmatrix}$$

The conservation laws in the case of the string are then:

$$\frac{\partial T_{tt}}{\partial t} + \frac{\partial T_{tx}}{\partial x} = 0$$

and similarly

$$\frac{\partial T_{xt}}{\partial t} + \frac{\partial T_{xx}}{\partial x} = 0$$

[15] We could consider more general cases, in which the Lagrangian density also depends explicitly on the space and time coordinates $t, x_0, \ldots, x_d$. I leave the generalization of the discussion to this case as an (optional) exercise.

In order to see what these laws mean physically, let us denote the energy in the piece of string lying between $x = a$ and $x = b$ by $E_{(a,b)}(t)$. Since we had that the energy density is given by T_{tt}, we have that

$$E_{(a,b)} = \int_a^b T_{tt}\, dx.$$

The energy in this piece of string will not be conserved. It might be at rest at one time, and then a few seconds later acquire energy as a wave passes between $x = a$ and $x = b$, and then later, lose all its energy as the wave passes on. How the energy in this portion of the string varies is given by

$$\begin{aligned}\frac{d}{dt}(E_{(a,b)}(t)) &= \frac{d}{dt}\int_a^b T_{tt}\, dx \\ &= \int_a^b \frac{\partial T_{tt}}{\partial t}\, dx \\ &= -\int_a^b \frac{\partial T_{tx}}{\partial x}\, dx \\ &= -\left[T_{tx}\right]_a^b \\ &= (T_{tx})_{x=a} - (T_{tx})_{x=b}\end{aligned}$$

where in going from the second to the third line we have used the conservation law. In this way, the rate of change in the energy in the interval (a, b) can be expressed in terms of the difference of a function evaluated at $x = a$ and $x = b$. If we interpret $T_{tx} = -\tau u_t u_x$ as the flux of energy moving from left to right, then our formula can be interpreted as the rate of change of energy of the string in the interval (a, b) is equal to the flux of energy coming into the segment of string from the left at $x = a$ minus the flux of energy leaving the string segment to the right at $x = b$.

Note that the the rate of change of E, the total energy on the whole string, is given by

$$\frac{dE}{dt} = \frac{d}{dt}\left(E_{(-\infty,\infty)}\right) = \tau\left[u_t u_x\right]_{-\infty}^{\infty}.$$

This rate of change vanishes, so that the total energy is conserved, provided that $u_t u_x \to 0$ as $|x| \to \infty$. In other words, the energy is conserved provided none of it leaks away at infinity. If we disturb the string at $t = 0$ near $x = 0$, it will take an infinite amount of time before the disturbance propagates out to infinity, so indeed energy will be conserved.