§5.7 Strings with Boundaries

Now that we know how to deal with infinitely long strings which run from $x = -\infty$ to $x = \infty$, let us complicate the situation a bit by introducing a boundary, or end, to our string at $x = 0$. The string is still infinitely long but now runs from $x = -\infty$ to $x = 0$. In such a situation it is necessary to specify a boundary condition at $x = 0$, specifying how the string interacts with the boundary. The most natural thing that we can impose is that no energy flows into the boundary. This is what one should expect if the string is attached to a rigid boundary of infinite mass: in this (idealised) case the vibrations of the string do not affect the boundary at all, and in particular there is no energy flow into the boundary.

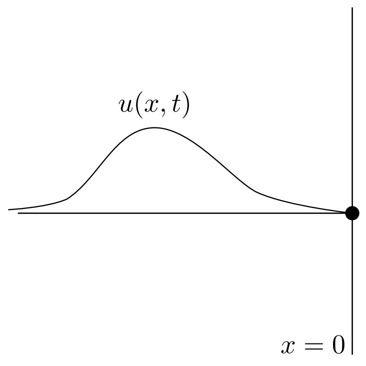
We have seen above that the right-moving energy flux for the string is $T_{tx} = -\tau u_x u_t$. So the condition that no energy flows into the boundary is

$$\lim_{x\to 0^-} T_{tx}(x,t) = -\lim_{x\to 0^-} \tau u_x(x,t) u_t(x,t) = 0\,.$$

There are two natural solutions to this equation: $\lim_{x\to 0^-} u_t(x,t) = 0$ and $\lim_{x\to 0^-} u_x(x,t) = 0$. For convenience, at the cost of some slight imprecision, we will refer to these conditions as $u_t(0,t) = 0$ and $u_x(0,t) = 0$. We study them in turn.

§5.7.1 Dirichlet boundary condition

The first case, $u_t(0,t) = 0$ is perhaps the most natural: it enforces that the endpoint of the string at $x = 0$ does not change with time, or in other words $u(0,t)$ is a constant. This is what you get if you simply tie a string to a wall. Given that there is a shift symmetry for u, let us simply assume that the condition is that $u(0,t) = 0$. This is called a *Dirichlet* boundary condition. It is quite straightforward to find the general solution in this case.



We know that $u(x,t)$ satisfies the wave equation for $x < 0$, so the solution must be of D'Alembert's form

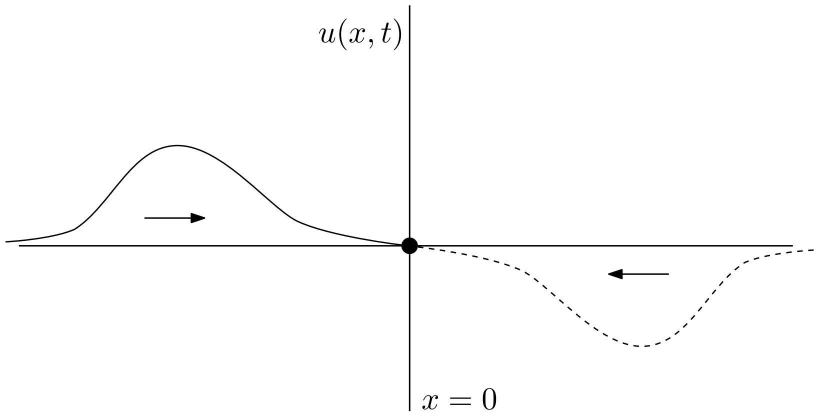
$$u(x,t) = f(x - ct) + g(x + ct) = f(x - ct) + h(-x - ct)$$

where for convenience we have introduced a function $h(\xi) = g(-\xi)$. The boundary condition tells us that

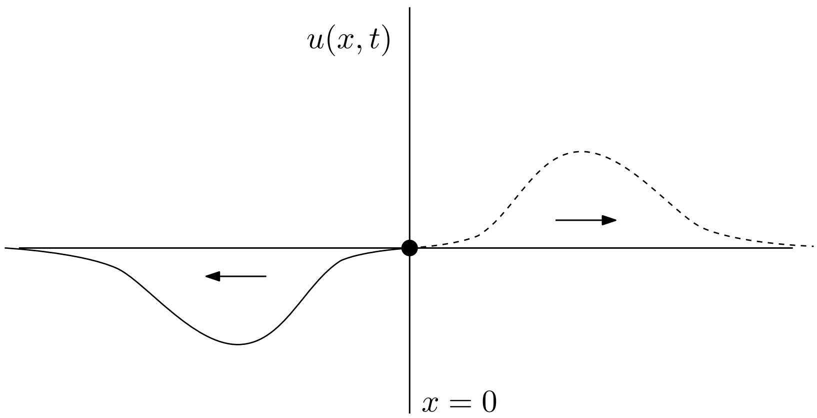
$$u(0,t) = 0 = f(-ct) + h(-ct),$$

from which it follows that $h(\xi) = -f(\xi)$. It follows that $u(x,t) = f(x-ct) - f(-x-ct)$.

To understand this solution a little better, note that, considered as a function on the whole of the x-axis, $u(x,t)$ is an odd function in x; that is $u(x,t) = -u(-x,t)$. The figure



shows the solution $u(x,t)$ for all x. In the physical region there is a wave moving towards the boundary. The dotted line represents a mirror image of the physical string. This mirror image moves to the left, and after some time will pass the line $x = 0$, emerging into the physical region $x < 0$ as the reflected wave. At later times the solution will look like below.



So we see from this that waves reflect off the boundary and are turned upside down by this boundary condition.

§5.7.2 Neumann boundary condition

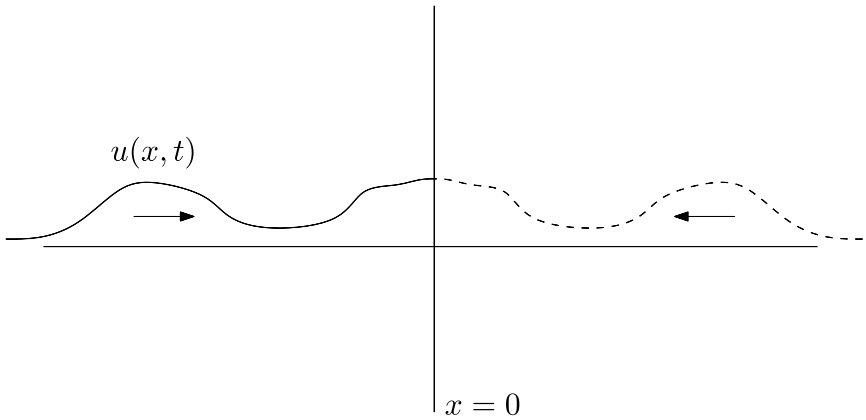
The other classic boundary condition for a string is the *Neumann* (sometimes called *free*) boundary condition $u_x(0,t) = 0$. Again the flux of energy into the boundary vanishes, so that energy is conserved on the string. Once more we can deduce the general solution from D'Alembert's solution $u(x,t) = f(x - ct) + h(-x - ct)$. Demanding that $u_x(0,t) = 0$ gives us that

$$u_x(0,t) = f'(0 - ct) - h'(0 - ct) = 0$$

from which we deduce that it is possible to take $f(\xi) = h(\xi)$ (up to a constant shift of u), so that

$$u(x,t) = f(x - ct) + f(-x - ct).$$

In this case, the function $u(x,t)$ considered over the whole line is an even function. As



before, given enough time the mirror image of the incoming wave emerges from behind the boundary $x = 0$ as the reflected wave, but in this case since $u(x,t)$ is even rather than odd it will emerge the same way up as the incoming wave.