§5.8 Junctions

Junctions or defects afford another possible way of introducing boundary conditions. We shall explain the idea of junctions through an example.

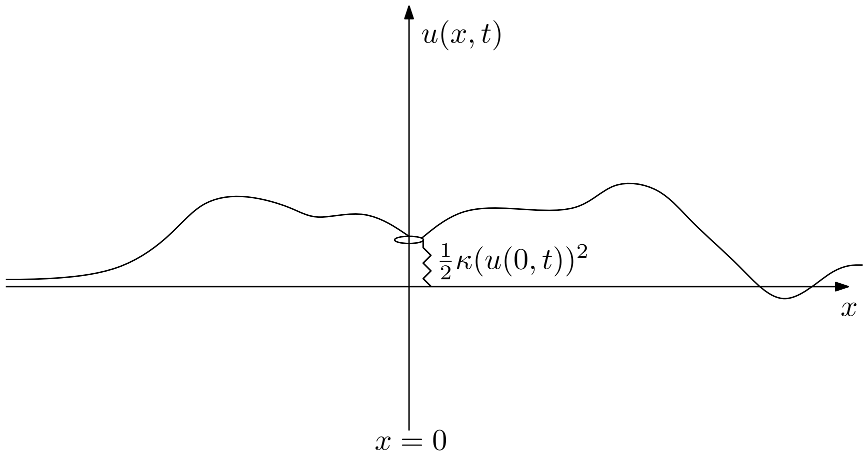

Figure 3: String with a spring of constant κ attached at $x = 0$.

Consider a setup in which we attach at $x = 0$ a spring, with constant κ and zero natural length, to the string, as in figure 3. We can view this system as two strings, one on the right and another on the left, joined at a junction at $x = 0$. Away from the junction at $x = 0$ we have a vanilla string, so we expect the monochromatic wave to be a good solution there. We want to understand what happens to such a monochromatic wave coming from the left as it hits the junction. Physically, we expect that part of the wave will be transmitted across the junction, and part will be reflected.

In order to solve the problem, it is essential to introduce *junction conditions*, describing which conditions should u satisfy as we cross the junction. The first condition is straightforward, namely that u is continuous at $x = 0$:

$$\lim_{\epsilon\to 0^+} u(\epsilon, t) = \lim_{\epsilon\to 0^-} u(\epsilon, t) \tag{5.8.1}$$

The second condition is energy conservation across the junction. In order to formulate this, note that on an infinitesimal neighbourhood $[-\epsilon, \epsilon]$ of $x = 0$ we have the energy

$$\frac{1}{2}\kappa\, u(0,t)^2 + \int_{-\epsilon}^{+\epsilon} dx \left(\frac{1}{2}\rho(u_t)^2 + \frac{1}{2}\tau(u_x)^2\right) .$$

That is, there is a contribution coming from the vibrating string between ϵ and ϵ, and a contribution from the extended spring at $x = 0$. We will assume that

$$\lim_{\epsilon\to 0} \int_{-\epsilon}^{+\epsilon} dx \left(\frac{1}{2}\rho(u_t)^2 + \frac{1}{2}\tau(u_x)^2\right) = 0\,.$$

so the only contribution to the total energy of the small interval in the limit $\epsilon \to 0$ is the one coming from the extension of the spring. Conservation of energy tells us

$$\frac{d}{dt}\left(\lim_{\epsilon\to 0} E(-\epsilon,\epsilon)\right) = \lim_{\epsilon\to 0}(T_{tx})_{x=-\epsilon} - \lim_{\epsilon\to 0}(T_{tx})_{x=\epsilon}\,. \tag{5.8.2}$$

As an example, suppose that we send in a monochromatic wave of unit amplitude. We expect that upon encountering the spring, this will be partially reflected into a left moving wave on the left side of the string, and partially transmitted to a right moving wave on the right side of the string. Putting this together our ansatz is

$$u(x,t) = \begin{cases} \Re\left(\left(e^{ipx} + Re^{-ipx}\right)e^{-ipct}\right) & \text{for } x \leq 0 \\ \Re\left(Te^{ip(x-ct)}\right) & \text{for } x > 0 \end{cases}$$

where T gives the amplitude/phase of the transmitted wave. Away from $x = 0$ we have monochromatic waves, which satisfy the wave equation. All that remains is to ensure that the ansatz also satisfies the junction conditions, by adjusting R and T. Continuity of $u(x,t)$ at $x = 0$ — that is, equation (5.8.1) — implies

$$\Re((1+R)e^{-ipct}) = \Re(Te^{-ipct})\,.$$

This will hold for all t if and only if[16]

$$1 + R = T\,.$$

This is our first junction condition in this case.

In order to study energy conservation, as given by equation (5.8.2), it is convenient to note that for our monochromatic wave solution continuity of $u(x,t)$ at $x = 0$ (or equivalently $1 + R = T$, as we just showed) implies

$$\lim_{x\to 0^-} u_t(x,t) = \lim_{x\to 0^+} u_t(x,t)\,.$$

In other words, $u_t(x,t)$ is continuous at $x = 0$, and $u_t(0,t)$ is well defined.

We have computed above that $T_{tx} = -\tau u_x u_t$, and we have that $\lim_{\epsilon\to 0} E(-\epsilon,\epsilon) = \frac{1}{2}\kappa\, u(0,t)^2$, so in the current case energy conservation across the junction becomes

$$\kappa\, u(0,t)u_t(0,t) = \tau \left[u_t u_x\right]_{-\epsilon}^{\epsilon} \tag{5.8.3}$$

This is our second junction condition. This equation can be simplified since there is a factor of $u_t(0,t)$ on both sides that we can divide, to obtain:

$$\kappa u(0,t) = \tau\left[\lim_{x\to 0^+} u_x(x,t) - \lim_{x\to 0^-} u_x(x,t)\right]\,.$$

[16] To see this, choose for instance $t = 0$ and $t = \pi/(2pc)$.

Plugging in our candidate monochromatic solution, this is

$$\kappa\Re((1+R)e^{-ipct}) = \tau\Re\left(ip(T-(1-R))e^{-ipct}\right)$$

which holds for all t if and only if

$$\kappa(1+R) = i\tau p(R+T-1)\,.$$

Solving this equation together with the continuity equation $1+R=T$ we find that

$$R = \frac{\kappa}{2ip\tau - \kappa}$$
$$T = \frac{2ip\tau}{2ip\tau - \kappa}$$

To get some intuition for these formulas, first assume that we make the spring very stiff by sending $\kappa \to \infty$. Then $R \to -1$ and $T \to 0$. This is as we expect; if the spring becomes very stiff, then the left hand piece of string has its end effectively pinned so it has a Dirichlet boundary condition, and nothing gets through to the right hand side. On the other hand if we send $\kappa \to 0$, then we are effectively removing the spring, and the two pieces of string will become as one. In this case we can explicitly see that as $\kappa \to 0$, $R \to 0$ and $T \to 1$.

Alternatively, we can think of fixing κ and consider the effect on waves with different value of p. The energy flux associated with the incoming wave, whose amplitude is fixed at one, is given by

$$\frac{\tau c p^2 |A|^2}{2} = \frac{\tau c p^2}{2}.$$

If p is very small, that is to say the wavelength is very long, then the energy flux is very small, and again it is hard for the wave to excite the spring since it does not have enough energy, so effectively we have Dirichlet boundary conditions. Again in this limit $p \to 0$, $R \to -1$ and $T \to 0$, the value for Dirichlet boundary conditions. On the other hand, if p is very large, the energy of the incoming wave is so large that the spring has little effect, and indeed $R \to 0$, $T \to 1$ in this limit.