§6 The Hamiltonian formalism

§6.1 Phase space

So far we have discussed the Lagrangian formalism, in which the evolution of the system is determined by the Euler-Lagrange equations. Given a set of initial conditions, these equations determine the time evolution of the system in configuration space. Recall from §2.2 that this is the space described by the generalized coordinates $\mathbf{q}$, without including the information about the velocities $\dot{\mathbf{q}}$.

The *Hamiltonian formalism* is closely related to the Lagrangian formalism that we have been studying so far, but it starts from a slightly different perspective: instead of considering configuration space, we now want to consider the space of all states of our physical systems. This space is known as *phase space*. I now define these notions.

Definition 6.1.1. The *state* of a classical system at a given instant in time is a complete set of data that fully fixes the future evolution of the system.

Example 6.1.2. *The Euler-Lagrange equations are second order linear differential equations on $\mathbf{q}(t)$ (assuming that the Lagrangian depends on $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ only, and not higher derivatives of $\mathbf{q}(t)$). We can fix the integration constants that appear in solving these equations by giving the positions $\mathbf{q}(t_0)$ and velocities $\dot{\mathbf{q}}(t_0)$ at any chosen time t_0, for some convenient choice of generalised coordinates and velocities. Once we have fixed these constants we know the behaviour of the system for all future times, so in this case we can parametrize the state at a given time t by giving $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$.*

Remark 6.1.3. The parametrization of the physical state in terms of $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ is not the only possible one: any parametrisation that allows us to fully fix future evolution is valid. We will see an example of a different parametrisation momentarily.

Definition 6.1.4. The *phase (or state) space* $\mathscr{P}$ of a classical system is the space of all possible states that the system can be in at a given instant in time.

Remark 6.1.5. This definition for phase space sounds rather similar to the definition of configuration space (this was definition 2.2.1). But note that that phase space has *twice* the dimension of configuration space: while configuration space encodes the (generalised) position of the system at a time t, phase space encodes the generalised positions *and* the velocities.

Example 6.1.6. *Consider a particle moving in one dimension. Phase space in this case is the two dimensional plane $\mathbb{R}^2$: one coordinate for x and one coordinate for $\dot{x}$. Every possible point in this plane is a possible state for the particle. For instance, the point $(x, \dot{x}) = (0, 10)$ parametrizes a particle at the origin, moving toward positive values of x. Similarly the particle moving in d dimensions has a phase space $\mathbb{R}^{2d}$. Note that the precise* *form of the Lagrangian does not enter in our definition of phase space: given a point in phase space the Lagrangian will determine future evolution, but any point in phase space is acceptable as an initial condition (by definition).*

Definition 6.1.7. The *Hamiltonian formalism* studies dynamics on phase space, parametrized by generalised coordinates $\mathbf{q}(t)$ and their associated generalised momenta $\mathbf{p}(t)$.

The fundamental step in going from the Lagrangian to the Hamiltonian formalism is to invert the definition equations for the generalised momenta:

$$p_i := \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_i} .$$

The right hand side of these equations are a set of functions of $\mathbf{q}$, $\dot{\mathbf{q}}$ and t. We often[17] can invert these equations to express $\dot{\mathbf{q}}$ in terms of $\mathbf{q}$, $\mathbf{p}$ and t. Once we do this, we can express any function in phase space (the Lagrangian, for instance) in terms of $\mathbf{q}$, $\mathbf{p}$ and t only.

Example 6.1.8. *Consider a particle of mass m moving in one dimension, expressed in Cartesian coordinates. Its Lagrangian is*

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 ,$$

so its associated momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} .$$

We can trivially solve this equation to find $\dot{x} = p/m$. We find that the Lagrangian for this system is thus

$$L(x, p) = \frac{p^2}{2m}$$

in the Hamiltonian formalism.

Example 6.1.9. *Let us now take a particle moving in two dimensions, expressed in polar coordinates. Its Lagrangian is*

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)$$

so its generalised momenta are

$$p_r = m\dot{r} \qquad ; \qquad p_\theta = mr^2\dot{\theta} .$$

We can easily invert these equations, to find $\dot{r} = p_r/m$ and $\dot{\theta} = p_\theta/(mr^2)$. In this way we can express any function of phase space in terms of the $\mathbf{q}$ and $\mathbf{p}$. For instance, for the Lagrangian itself we have

$$L(\mathbf{q}, \mathbf{p}) = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) .$$

[17] This will be true in the examples that we discuss during this course, at any rate. There are interesting situations in which this inversion cannot be done, but we will not study them during this course. I encourage those of you who are curious to search for material on "Dirac brackets" if you want to see how our story below generalises to these more complicated cases. A good reference is the book *"Quantization of Gauge Systems"*, by Henneaux and Teitelboim.