§6.2 Example: the wave equation from the Lagrangian for a string

Our main example will be a Lagrangian density that can be thought of the Lagrangian density for the one-dimensional string oscillating in one dimension. The standard name for this Lagrangian is the "massless scalar field" Lagrangian.

Definition 6.2.1. The *massless scalar field Lagrangian* is

$$\mathcal{L} = \frac{1}{2}\rho u_t^2 - \frac{1}{2}\tau u_x^2 \,.$$

We refer to the constants ρ and τ as the *density* and *tension*, respectively. The field "u" in this expression is the *massless scalar*.

Remark 6.2.2. It is in fact possible, and we do this in section 6.2.1 below, to derive this Lagrangian density from the physics of an idealized string in the limit in which the oscillations are small. This explains the origin of the labels "density" and "tension" above. I emphasize that the uses of this Lagrangian in Mathematical Physics go well beyond explaining vibrating strings.

Definition 6.2.3. The Euler-Lagrange equations for fields immediately imply the equation of motion

$$\rho u_{tt} - \tau u_{xx} = 0$$

for the massless scalar u, where

$$u_{tt} := \frac{\partial^2 u}{\partial t^2} = \frac{\partial u_t}{\partial t}$$

and similarly for u_{xx}. Introducing for convenience $c^2 = \tau/\rho$ (both the tension and the density are assumed to be positive, so c is real), the equation of motion for the massless scalar becomes:

$$u_{tt} = c^2 u_{xx} \,.$$

We will refer to this equation as the *wave equation*. More precisely, what we are describing here is known as the wave equation in one spatial dimension.

§6.2.1 Derivation of the massless scalar Lagrangian from a physical system

ℹ *This section is* ***not examinable****.* ℹ

We will now derive the massless scalar Lagrangian from the dynamics of a string vibrating in one dimension, in the approximation where the displacements are small. Similarly to the case of point particles, the Lagrangian density can be constructed in terms of the kinetic and potential energy densities. That is, if we have

$$T(u, u_x, u_t, x, t) = \int dx\, \mathcal{T}(u, u_x, u_t, x, t)$$

and

$$V(u, u_x, u_t, x, t) = \int dx\, \mathcal{V}(u, u_x, u_t, x, t)$$

for the total kinetic energy T and total potential energy V of the string, then we call $\mathcal{T}$ and $\mathcal{V}$ the corresponding *densities* of kinetic and potential energy, and we have

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

So we need to find expressions for the kinetic and potential energy densities. We will work to leading (that is, quadratic) order in u_x and u_t. This is the regime in which the oscillations are neither too large nor too fast. We do this because it leads to much simpler equations, while still being quite useful for modelling many systems in Nature. Similarly, we will assume that the string is only displaced vertically, without any horizontal displacement.

The kinetic energy can be obtained relatively straightforwardly by subdividing the string into small pieces. Consider the small piece lying between x and $x + \delta x$. If the segment is small enough its behaviour will be approximately point-like; therefore its kinetic energy will be of the form $\frac{m}{2}v^2$. The mass of the small segment of string is given by

$$m = \rho\, ds \approx \rho\sqrt{1 + (u_x)^2}\,\delta x \approx \rho\,\delta x.$$

Here ρ is the density of the string (which we take to be constant), and ds the arc-length of the string segment. The final approximation follows from taking $u_x \ll 1$. Since $u(x,t)$ denotes the vertical displacement of the string it is clear that the vertical velocity is u_t. The contribution to the kinetic energy from the small piece of string that we are considering is then $\frac{1}{2}(u_t)^2\rho\,\delta x$. We then immediately obtain the kinetic energy of the whole string by integrating over all the segments to find that the kinetic energy is given by

$$T = \frac{\rho}{2}\int_{-\infty}^{\infty} dx\,(u_t)^2$$

so the kinetic energy *density* is

$$\mathcal{T} = \frac{\rho}{2}(u_t)^2\,.$$

Obtaining the potential energy is a little bit more subtle. We know that the tension in the string is a constant, which we call τ. It follows that the work done in extending the string's length by a distance δl will be $\tau\delta l$. If we imagine extruding the entire length of the string from a point we reach the conclusion that the potential energy of the string is τ times its length. Of course, our string is infinitely long, so that this may initially be a concern, until we recall that adding a constant to the potential energy makes no difference. We are not really interested in the absolute value of the potential energy, but rather the differences in potential energy between string in various configurations. Therefore we will take the potential energy of a string in some configuration $u(x,t)$ to be defined as τ times the difference in length between the string with shape $u(x,t)$ and the length of the undisturbed string lying along the x-axis for which $u(x,t) = 0$. To be more precise we have

$$\begin{aligned} V &= \tau\left(\int_{-\infty}^{\infty} ds - \int_{-\infty}^{\infty} dx\right) \\ &= \tau\left(\int_{-\infty}^{\infty} (\sqrt{1+(u_x)^2} - 1)dx\right) \\ &\approx \tau\left(\int_{-\infty}^{\infty} (1 + \frac{(u_x)^2}{2} - 1)dx\right) \\ &= \frac{\tau}{2}\int_{-\infty}^{\infty} (u_x)^2\, dx \end{aligned}$$

again to leading order in oscillations. From here we obtain the potential energy density

$$\mathcal{V} = \frac{\tau}{2}(u_x)^2$$

and thus the Lagrangian density

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \frac{\rho}{2}(u_t)^2 - \frac{\tau}{2}(u_x)^2\,.$$