§6.2 The Poisson bracket and Hamiltonian flows

We still need to understand how a given state evolves in time in this new formalism. That is, if we know which point in phase space describes a system at a given time, which trajectory in phase space will describe subsequent motion of the system?

In fact, there would be little point in doing this if all we gained was a description of the dynamics in a different set of variables. After all, the Lagrangian formalism will do the job of giving the equations of motion for the system perfectly well.[18] The advantage of switching to the Hamiltonian formalism is that we will be able to exhibit a rather deep and beautiful geometric structure to classical dynamics, in which we will obtain (in a sense) a reciprocal of Noether's theorem! Recall that Noether's theorem states that every symmetry has an associated conserved charge. We will see below that in the Hamiltonian formalism the conserved charge *generates* the symmetry: if we know the form of the conserved charge for a symmetry we will be able to reconstruct systematically the infinitesimal form of the symmetry transformation.

The fundamental object that allows us to think of charges as generating transformations is the Poisson bracket:

Definition 6.2.1. The *Poisson bracket* between two functions $f(\mathbf{q}, \mathbf{p}, t)$ and $g(\mathbf{q}, \mathbf{p}, t)$ on phase space is the function in phase space defined by

$$\{f, g\} := \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{6.2.1}$$

where n is the dimension of configuration space (so half the dimension of phase space).

Remark 6.2.2. Note that in the definition of the Poisson bracket the position and momenta are independent coordinates in phase space, and are treated as independent variables when taking partial derivatives:

$$\frac{\partial q_i}{\partial p_j} = \frac{\partial p_i}{\partial q_j} = 0 \qquad ; \qquad \frac{\partial q_i}{\partial q_j} = \frac{\partial p_i}{\partial p_j} = \delta_{ij} \,.$$

Example 6.2.3. *The simplest functions in phase space that we can construct are those that give the coordinates of a point in a given basis. From the definition of the Poisson bracket, we have the fundamental brackets*

$$\{q_i, q_j\} = \{p_i, p_j\} = 0 \qquad ; \qquad \{q_i, p_j\} = \delta_{ij} \,.$$

[18] Or Newton's formalism, for that matter! We went through all this trouble during the past weeks not because we wanted to find more efficient methods of solving the dynamics of classical systems (although that is sometimes a useful byproduct of switching perspectives), but rather because we wanted to understand better the structure of classical mechanics — important ideas like the action principle or the relation between symmetries and conserved charges become much more transparent in the Lagrangian and Hamiltonian formalisms.

The Poisson bracket has a number of interesting properties, which I now list. The proof of these properties is straightforward, and can be found in the problem sheet for week 10:

Proposition 6.2.4. *The Poisson bracket is* ***antisymmetric****:*

$$\{f, g\} = -\{g, f\}\,.$$

Proposition 6.2.5. *The Poisson bracket is* ***linear****:*

$$\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$$

for $\alpha, \beta \in \mathbb{R}$*. Note that together with antisymmetry this implies*

$$\{h, \alpha f + \beta g\} = \alpha\{h, f\} + \beta\{h, g\}$$

so the Poisson bracket is in fact ***bilinear*** *(that is, linear on both terms).*

Proposition 6.2.6. *The Poisson bracket obeys the* ***Leibniz identity****:*

$$\{fg, h\} = f\{g, h\} + g\{f, h\}\,.$$

Proposition 6.2.7. *The Poisson bracket obeys the* ***Jacobi identity*** *for the sum of the cyclic permutations:*

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0\,.$$