Denote by $\mathscr{F}$ the space of all functions from phase space $\mathscr{P}$ to $\mathbb{R}$. Given any function $f \in \mathscr{F}$, we can define an operator $\Phi_f$ that generates infinitesimal transformations on $\mathscr{F}$ using the Poisson bracket.

**Definition 6.2.8.** The *Hamiltonian flow* defined by $f\colon \mathscr{P} \to \mathbb{R}$ is the infinitesimal transformation on $\mathscr{F}$ defined by

$$\Phi_f^{(\epsilon)}\colon \mathscr{F} \to \mathscr{F}$$
$$\Phi_f^{(\epsilon)}(g) = g + \epsilon\{g, f\} + \mathcal{O}(\epsilon^2)\,.$$

*Remark* 6.2.9. I am taking a small liberty with the language here to avoid having to introduce some additional formalism: what I have just introduced is the infinitesimal version of what is commonly known as "Hamiltonian flow" in the literature, which is typically defined for finite (that is, non-infinitesimal) transformations. The finite version of the transformation is obtained by exponentiation:

$$\Phi_f^{(a)}(g) = e^{a\{\cdot,f\}}g \coloneqq g + a\{g, f\} + \frac{a^2}{2!}\{\{g, f\}, f\} + \frac{a^3}{3!}\{\{\{g, f\}, f\}, f\} + \dots$$

*Remark* 6.2.10. By studying the action of $\Phi_f^{(\epsilon)}$ on the coordinates $\mathbf{q}$, $\mathbf{p}$ of phase space, we can also understand $\Phi_f^{(\epsilon)}$ as the generator of a map from phase space to itself. We have

$$\Phi_f^{(\epsilon)}(q_i) = q_i + \epsilon\{q_i, f\} + O(\epsilon^2) = q_i + \epsilon\frac{\partial f}{\partial p_i} + O(\epsilon^2)$$
$$\Phi_f^{(\epsilon)}(p_i) = p_i + \epsilon\{p_i, f\} + O(\epsilon^2) = p_i - \epsilon\frac{\partial f}{\partial q_i} + \mathcal{O}(\epsilon^2)\,.$$

The two definitions are compatible:

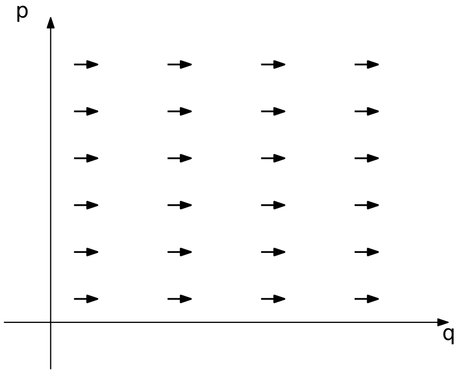
$$\begin{aligned}
\Phi_f^{(\epsilon)}(g) &= g(q_1 + \epsilon\{q_1, f\}, \dots, q_n + \epsilon\{q_n, f\}, p_1 + \epsilon\{p_1, f\}, \dots, p_n + \epsilon\{p_n, f\}) \\
&= g(q_1, \dots, q_n, p_1, \dots, p_n) + \epsilon\sum_{i=1}^{n}\left(\frac{\partial g}{\partial q_i}\{q_i, f\} + \frac{\partial g}{\partial p_i}\{p_i, f\}\right) \\
&= g(q_1, \dots, q_n, p_1, \dots, p_n) + \epsilon\sum_{i=1}^{n}\left(\frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right) \\
&= g + \epsilon\{g, f\}
\end{aligned}$$

where in the second line we have done a Taylor expansion, and we have omitted higher order terms in $\epsilon$ throughout for notational simplicity.

**Example 6.2.11.** *As a simple example, consider a particle moving in one dimension. The Hamiltonian flow $\Phi_p$ associated to the canonical momentum $p$ acts on phase space functions as:*

$$\Phi_p^{(\epsilon)}(g(q, p)) = g(q, p) + \epsilon\frac{\partial g}{\partial q} + \mathcal{O}(\epsilon^2)\,.$$

*Alternatively, $\Phi_p^{(\epsilon)}$ acts on the coordinate $q$ as $q \to q + \epsilon$, so the effect of $\Phi_p^{(\epsilon)}$ on phase space is a uniform shift in the $q$ direction:*




*We can reproduce the effect on arbitrary functions of $q$ from this viewpoint by doing a Taylor expansion:*

$$g(q+\epsilon, p) = g(q,p) + \epsilon \frac{\partial g}{\partial q} + \mathcal{O}(\epsilon^2)\,.$$

*(You might also find it interesting to reproduce the full form of the Taylor expansion of $f(x+a)$ around $x$ using the exponentiated version in remark 6.2.9.)*

**Example 6.2.12.** *As a second example, consider a particle of unit mass moving in two dimensions, expressed in Cartesian coordinates, which we call $q_1$ and $q_2$. We choose the Lagrangian to be of the form*

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1, q_2)\,.$$

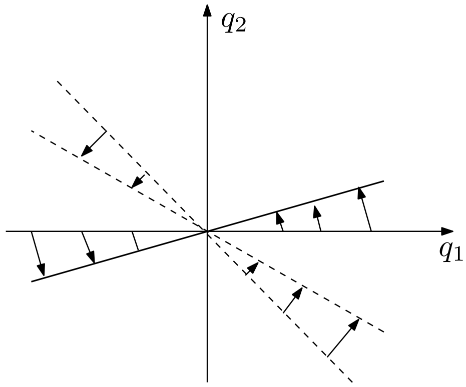
*For the function generating the flow we will choose $J = q_1\dot{q}_2 - q_2\dot{q}_1$. (Recall from example 3.1.16 that this function is angular momentum, which Noether's theorem associated with rotations around the origin.) From the Lagrangian we have $p_1 = \dot{q}_1$ and $p_2 = \dot{q}_2$, so in terms of standard $(\mathbf{q}, \mathbf{p})$ coordinates of phase space we have $J(\mathbf{q}, \mathbf{p}) = q_1p_2 - q_2p_1$. The Hamiltonian flow $\Phi_J^{(\epsilon)}$ then acts on phase space as*

$$\begin{aligned}
\Phi_J^{(\epsilon)}(q_1) &= q_1 + \epsilon\{q_1, J\} = q_1 + \epsilon\frac{\partial J}{\partial p_1} = q_1 - \epsilon q_2\,,\\
\Phi_J^{(\epsilon)}(q_2) &= q_2 + \epsilon\{q_2, J\} = q_2 + \epsilon\frac{\partial J}{\partial p_2} = q_2 + \epsilon q_1\,,\\
\Phi_J^{(\epsilon)}(p_1) &= p_1 + \epsilon\{p_1, J\} = p_1 - \epsilon\frac{\partial J}{\partial q_1} = p_1 - \epsilon p_2\,,\\
\Phi_J^{(\epsilon)}(p_2) &= p_2 + \epsilon\{p_2, J\} = p_2 - \epsilon\frac{\partial J}{\partial q_2} = p_2 + \epsilon p_1\,.
\end{aligned}$$

*omitting higher orders in $\epsilon$. So the effect of $J$ on the coordinates can be written as an infinitesimal rotation on the $\mathbf{q}$ and the $\mathbf{p}$ (independently)*

$$\begin{aligned}
\Phi_J^{(\epsilon)}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\,,\\
\Phi_J^{(\epsilon)}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} &= \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\,.
\end{aligned}$$

*For instance, the action of* $\Phi_J^{(\epsilon)}$ *on the* $(q_1, q_2)$ *slice of phase space (which in this case has four dimensions) is as in the following picture:*




### *§6.2.1 Flows for conserved charges*

We have just seen that linear momentum $p$ generates spatial translations, and angular momentum generates rotations. This is in fact general: assume that we have a transformation acting as $q_i \to q_i + \epsilon a_i(\mathbf{q}) + O(\epsilon^2)$ on the generalised coordinates. Noether's theorem assigns a charge to this transformation given, in the Lagrangian framework, by

$$Q(\mathbf{q}, \dot{\mathbf{q}}, t) = \left( \sum_{i=1}^{n} a_i(\mathbf{q}) \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}_i} \right) - F(\mathbf{q}, t)\,.$$

This charge can be written in the Hamiltonian framework in terms of generalised coordinates and generalised momenta as

$$Q(\mathbf{q}, \mathbf{p}, t) = \left( \sum_{i=1}^{n} a_i(\mathbf{q}) p_i \right) - F(\mathbf{q}, t)\,.$$

If we now compute the Hamiltonian flow associated to this charge on the generalised coordinates we find

$$\Phi_Q^{(\epsilon)}(q_i) = q_i + \epsilon\{q_i, Q\} + O(\epsilon^2) = q_i + \epsilon a_i + \mathcal{O}(\epsilon^2)\,.$$

**Note 6.2.13**

This is a very important result: Noether's theorem told us that symmetries imply the existence of conserved quantities. We have just seen that we can go in the other direction too: conserved quantities *generate* the corresponding symmetry transformations, via the associated Hamiltonian flow.