

§6.3 D'Alembert's Solution to the Wave Equation

The general solution to the wave equation in one spatial dimension was given by *D'Alembert*, and it is simply

$$u(x, t) = f(x - ct) + g(x + ct)$$

where f and g are arbitrary functions. The part of the solution $f(x - ct)$ corresponds to a wave moving to the right with speed c , whilst the remaining part $g(x + ct)$ corresponds to a wave moving to the left with speed c .

Theorem 6.3.1. *D'Alembert's solution $u(x, t) = f(x - ct) + g(x + ct)$ is the general solution to the wave equation.*

Proof. We introduce new variables $x_+ = x + ct$ and $x_- = x - ct$, or equivalently $x = \frac{1}{2}(x_+ + x_-)$ and $t = \frac{1}{2c}(x_+ - x_-)$. By the Chain Rule:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x_+} \frac{\partial x_+}{\partial x} + \frac{\partial u}{\partial x_-} \frac{\partial x_-}{\partial x} = \frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-}, \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x_+} \frac{\partial x_+}{\partial t} + \frac{\partial u}{\partial x_-} \frac{\partial x_-}{\partial t} = c \left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-} \right). \end{aligned}$$

Taking derivatives again, once more using the Chain Rule:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-} \right) = \frac{\partial}{\partial x_+} \left(\frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-} \right) + \frac{\partial}{\partial x_-} \left(\frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-} \right) \\ &= \frac{\partial^2 u}{\partial x_+^2} + \frac{\partial^2 u}{\partial x_-^2} + 2 \frac{\partial^2 u}{\partial x_+ \partial x_-} \\ \frac{\partial^2 u}{\partial t^2} &= c \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-} \right) = c^2 \frac{\partial}{\partial x_+} \left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-} \right) - \frac{\partial}{\partial x_-} \left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-} \right) \\ &= c^2 \left(\frac{\partial^2 u}{\partial x_+^2} + \frac{\partial^2 u}{\partial x_-^2} - 2 \frac{\partial^2 u}{\partial x_+ \partial x_-} \right) \end{aligned}$$

As usual, we have used the assumption that partial derivatives commute. We see that in these variables the wave equation $u_{tt} = c^2 u_{xx}$ becomes

$$u_{tt} - c^2 u_{xx} = -4c^2 \frac{\partial^2 u(x_+, x_-)}{\partial x_+ \partial x_-} = 0.$$

The general solution of this equation is indeed

$$u(x_+, x_-) = f(x_-) + g(x_+).$$

□

In practice, we are often interested in understanding what happens if we release a string from a given configuration. How does the string evolve? This is an initial value problem, which D'Alembert also solved in general. Assume that we are told that at $t = 0$ the string has profile $\psi(x)$, that is

$$u(x, 0) = \varphi(x)$$

and in addition we know with which speed the string is moving at that instant:

$$u_t(x, 0) = \psi(x).$$

In terms of f and g , which parametrise the general form of the solution, these equations are

$$f(x) + g(x) = \varphi(x)$$

and

$$-cf'(x) + cg'(x) = \psi(x).$$

This last equation can be integrated (formally) to give

$$g(x) - f(x) = d + \frac{1}{c} \int_{-\infty}^x ds \psi(s)$$

with d some unknown constant. We now have two equations for two unknowns, so solving for f and g we find

$$\begin{aligned} f(x) &= \frac{1}{2} \left(\varphi(x) - d - \frac{1}{c} \int_{-\infty}^x ds \psi(s) \right) \\ g(x) &= \frac{1}{2} \left(\varphi(x) + d + \frac{1}{c} \int_{-\infty}^x ds \psi(s) \right) \end{aligned}$$

so we finally find

$$\begin{aligned} u(x, t) &= f(x - ct) + g(x + ct) \\ &= \frac{\varphi(x - ct) + \varphi(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} ds \psi(s). \end{aligned}$$