§6.3 *The Hamiltonian and Hamilton's equations*

We have just proven that conserved quantities generate the corresponding symmetries. It is natural to guess at this point that energy will generate time evolution, via Hamiltonian flow. This is indeed the case.

Definition 6.3.1. The *Hamiltonian "H"* of a physical system is the energy expressed in terms of generalised coordinates and generalised momenta. That is:

$$H \coloneqq \left(\sum_{i=1}^{n} p_i \dot{q}_i(\mathbf{q}, \mathbf{p}, t) \right) - L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t), t)\,.$$

Example 6.3.2. *Consider the harmonic oscillator in one dimension, with Lagrangian*

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} \kappa x^2\,.$$

The generalised momentum is $p = m\dot{x}$, so the Hamiltonian for this system is

$$H = \frac{1}{2m} p^2 + \frac{1}{2} \kappa x^2\,.$$

Theorem 6.3.3. *The time evolution of the generalised coordinates and momenta is given by the Hamiltonian flow Φ_H:*

$$\Phi_H(q_i) = q_i(t+\epsilon) + O(\epsilon^2) \qquad ; \qquad \Phi_H(p_i) = p_i(t+\epsilon) + O(\epsilon^2)\,.$$

Equivalently (expanding $q_i(t+\epsilon) = q_i(t) + \epsilon \dot{q}_i(t) + \ldots$, and similarly for p_i):

$$\boxed{\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i} \qquad ; \qquad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}\,.} \tag{6.3.1}$$

These equations are known as ***Hamilton's equations of motion****.*

Proof. The first thing to do is to note that when we write the partial derivative $\frac{\partial A}{\partial q_j}$ in the Hamiltonian picture we mean differentiate A with respect to q_j keeping the other q's, any explicit time dependance in A, *and the p's* fixed. This should be contrasted with the Lagrangian picture, where differentiating with respect to q_j involved keeping the other q's time, and *and the $\dot{q}$'s fixed.* To highlight this point, in this proof I will write $\frac{\partial A}{\partial q_j}|_{\mathbf{p}}$ or $\frac{\partial A}{\partial q_j}|_{\dot{\mathbf{q}}}$ to clarify which set of variables are being held fixed when taking partial derivatives.[19]

[19] Not to overload notation too much, I will leave implicit the fact that we are also keeping fixed any explicit time parameters in A, unless explicitly stated otherwise.

Given this let us calculate the derivates of H with respect to q_j and p_j. we have

$$\begin{aligned}
\left.\frac{\partial H}{\partial q_j}\right|_{\mathbf{p}} &= \frac{\partial}{\partial q_j}\left(\sum_i p_i\dot{q}_i(q,p,t) - L(q,\dot{q}(q,p,t),t)\right)\Bigg|_{\mathbf{p}} \\
&= \sum_i p_i \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{\mathbf{p}} - \sum_i \left.\frac{\partial L}{\partial q_i}\right|_{\dot{\mathbf{q}}} \left.\frac{\partial q_i}{\partial q_j}\right|_{\mathbf{p}} - \sum_i \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\mathbf{q}} \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{\mathbf{p}} \\
&= \sum_i p_i \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{\mathbf{p}} - \left.\frac{\partial L}{\partial q_j}\right|_{\dot{\mathbf{q}}} - \sum_i \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\mathbf{q}} \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{\mathbf{p}} \\
&= \sum_i \left(p_i - \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\mathbf{q}}\right) \left.\frac{\partial \dot{q}_i}{\partial q_j}\right|_{\mathbf{p}} - \left.\frac{\partial L}{\partial q_j}\right|_{\dot{\mathbf{q}}} .
\end{aligned}$$

The first bracket in this expression is zero by the definition of p_i. Using the Euler-Lagrange equations we conclude that along a physical path

$$\left.\frac{\partial H}{\partial q_j}\right|_{\mathbf{p}} = -\left.\frac{\partial L}{\partial q_j}\right|_{\dot{\mathbf{q}}} = -\frac{d}{dt}\left(\left.\frac{\partial L}{\partial \dot{q}_j}\right|_{\mathbf{q}}\right) = -\dot{p}_j .$$

Similarly, calculating $\frac{\partial H}{\partial p_j}$ we find

$$\begin{aligned}
\left.\frac{\partial H}{\partial p_j}\right|_{\mathbf{q}} &= \frac{\partial}{\partial p_j}\left(\sum_i p_i\dot{q}_i(q,p,t) - L(q,\dot{q}(q,p,t),t)\right)\Bigg|_{\mathbf{q}} \\
&= \sum_i \frac{\partial p_i}{\partial p_j}\dot{q}_i + \sum_i p_i \left.\frac{\partial \dot{q}_i}{\partial p_j}\right|_{\mathbf{q}} - \sum_i \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\mathbf{q}} \left.\frac{\partial \dot{q}_i}{\partial p_j}\right|_{\mathbf{q}} \\
&= \sum_i \delta_{ij}\dot{q}_i + \sum_i \left(p_i - \left.\frac{\partial L}{\partial \dot{q}_i}\right|_{\mathbf{q}}\right) \left.\frac{\partial \dot{q}_i}{\partial p_j}\right|_{\mathbf{q}} \\
&= \dot{q}_j
\end{aligned}$$

again using the definition of p_i to show the last term vanishes. Note that we did not need to use the Euler-Lagrange equations to derive this last equation. Accordingly, in practice this equation generally just reproduces the result of inverting the definition of the generalised momentum in the Lagrangian formalism to express $\dot{\mathbf{q}}$ in terms of $\mathbf{q}$, $\mathbf{p}$ and t. □

Corollary 6.3.4. *The time evolution of any function $f(\mathbf{q},\mathbf{p})$ on phase space is generated by Φ_H:*

$$\frac{df}{dt} = \{f, H\} .$$

In the case that f depends explicitly on time, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\} .$$

Proof. The function f will depend on time through its explicit dependence on t, if any, and via its implicit dependence via $\mathbf{q}$ and $\mathbf{p}$, who themselves are functions of time. Using the Chain Rule we find

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right)\\ &= \frac{\partial f}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right)\\ &= \frac{\partial f}{\partial t} + \{f, H\}\,,\end{aligned}$$

where we have used Hamilton's equations in going to the second line. □

Remark 6.3.5. We can apply this corollary to give a very neat proof of conservation of energy: the energy, in the Hamiltonian formalism, is equal to the Hamiltonian itself. So we have that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t} \tag{6.3.2}$$

using the fact that the Poisson bracket is antisymmetric. So, if time does not appear explicitly in the expression for the Hamiltonian, then the Hamiltonian is conserved.

Remark 6.3.6. A small variation of this last equation is sometimes included as part of Hamilton's equations. From the definition of the Hamiltonian we have that

$$\begin{aligned}\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial t} &= \frac{\partial}{\partial t}\left(\left(\sum_{i=1}^{n}\dot{q}_i(\mathbf{q},\mathbf{p},t)p_i\right) - L(\mathbf{q},\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t),t)\right)\\ &= \left(\sum_{i=1}^{n}p_i\frac{\partial \dot{q}_i(\mathbf{q},\mathbf{p},t)}{\partial t}\right) - \frac{\partial L(\mathbf{q},\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t),t)}{\partial t}\,.\end{aligned}$$

Now note that L can have an explicit dependence on t through $\dot{\mathbf{q}}$, if $\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t)$ depends explicitly on time. Using the Chain Rule:

$$\frac{\partial L(\mathbf{q},\dot{\mathbf{q}}(\mathbf{q},\mathbf{p},t),t)}{\partial t} = \left.\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t}\right|_{\mathbf{q},\dot{\mathbf{q}}} + \left(\sum_{i=1}^{n}\left.\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_i}\right|_{\mathbf{q}}\frac{\partial \dot{q}_i(\mathbf{q},\mathbf{p},t)}{\partial t}\right)$$

so

$$\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial t} = -\left.\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t}\right|_{\mathbf{q},\dot{\mathbf{q}}} + \left(\sum_{i=1}^{n}\left(p_i - \left.\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial \dot{q}_i}\right|_{\mathbf{q}}\right)\frac{\partial \dot{q}_i(\mathbf{q},\mathbf{p},t)}{\partial t}\right)\,.$$

The second term vanishes due to the definition of the generalised momentum, so we conclude that

$$\boxed{\left.\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial t}\right|_{\mathbf{q},\mathbf{p}} = -\left.\frac{\partial L(\mathbf{q},\dot{\mathbf{q}},t)}{\partial t}\right|_{\mathbf{q},\dot{\mathbf{q}}}} \tag{6.3.3}$$

In particular, this makes (6.3.2) compatible with theorem 3.2.2.

Remark 6.3.7. More generally, assume that we have a function $Q(\mathbf{q}, \mathbf{p}, t)$ on phase space. We have that Q is conserved if

$$\frac{dQ}{dt} = \{Q, H\} + \frac{\partial Q}{\partial t} = 0\,.$$

In particular, if Q does not depend explicitly on time, we have that Q is conserved if and only if $\{Q, H\} = 0$. By antisymmetry of the Poisson bracket we can also read this condition as

$$\{H, Q\} = 0$$

which can be interpreted as saying that the Hamiltonian is left invariant, to first order in ϵ, by the transformation generated by Q:

$$\Phi_Q(H) = H + \epsilon\{Q, H\} + O(\epsilon^2) = H + O(\epsilon^2)\,.$$