Example 6.3.8. *A system whose Lagrangian is given by*

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{r^2}{2}.$$

We define the momenta to be

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = \dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} \end{aligned}$$

so that

$$\begin{aligned} \dot{r} &= p_r \\ \dot{\theta} &= \frac{p_\theta}{r^2} \end{aligned}$$

The Hamiltonian is given by

$$\begin{aligned} H &= p_r\dot{r} + p_\theta\dot{\theta} - \left(\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{r^2}{2}\right) \\ &= p_r^2 + p_\theta\left(\frac{p_\theta}{r^2}\right) - \frac{1}{2}\left({p_r}^2 + r^2\left(\frac{p_\theta}{r^2}\right)^2 - \frac{r^2}{2}\right) \\ &= \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{r^2}{2}. \end{aligned}$$

Hamilton's Equations of Motion tell us that

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = p_r \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2} \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{r^3} - r \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0. \end{aligned}$$

Note that the first two equations here simply reproduce the results of expressing the $\mathbf{q}$ *in terms of the* $\mathbf{p}$*'s. This is always the case when we derive the Hamiltonian system from a Lagrangian system like above. The last equation shows that* p_θ *is conserved as a result of the Hamiltonian being independent of* θ*. The concept of an ignorable coordinate goes over completely from the Lagrangian picture to the Hamiltonian picture. The real 'meat' of the dynamics is in the remaining equation for* $\dot{p}_r$*. Given that* p_θ *is a constant and that* $p_r = \dot{r}$ *it can be read as*

$$\ddot{r} = \frac{p_\theta^2}{r^3} - r.$$

Example 6.3.9. *Suppose we start instead with a Hamiltonian:*

$$H = \frac{p^2}{2} + xp.$$

Hamilton's equations are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = p + x \\ \dot{p} &= -\frac{\partial H}{\partial x} = -p. \end{aligned}$$

Solving the second equation, we have that $p = Ae^{-t}$. Substituting this into the first equation we find

$$\dot{x} - x = Ae^{-t}$$

which is a linear first order differential equation. Multiplying through by the integrating factor we find

$$\frac{d}{dt}\left(xe^{-t}\right) = Ae^{-2t}$$

which can be integrated to give $x = Ce^t - Ae^{-t}/2$.

Example 6.3.10. *The following is a Hamiltonian for the damped harmonic oscillator:*

$$H = \frac{e^{-bt}p^2}{2} + \frac{e^{bt}w^2x^2}{2}.$$

Notice that H explicitly depends on time; this implies that it is not conserved, as we would expect for the damped harmonic oscillator, whose motion dies away to nothing. Hamilton's equation of motion are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = e^{-bt}p \\ \dot{p} &= -\frac{\partial H}{\partial x} = -w^2e^{bt}x. \end{aligned}$$

Differentiating the first equation with respect to t, we see that

$$\begin{aligned} \ddot{x} &= -be^{-bt}p + e^{-bt}\dot{p} \\ &= -b\dot{x} - w^2x \end{aligned}$$