§6.4 There and back again

> ℹ *This section is **not examinable**.* ℹ

Let me finish by closing the circle of ideas that we have been developing. We have seen in §3.1 that Noether's theorem implies that every symmetry implies the existence of a conserved Noether charge Q, and we have also shown in §6.2.1 that the Noether charge associated to a symmetry generates the right transformations on the generalized coordinates. It is natural to ask at this point: does any conserved charge generate a symmetry transformation? This is indeed that case, as we now show for the class of conserved charges that we have been discussing.

Theorem 6.4.1. *Assume that we have a function $Q(\mathbf{q}, \mathbf{p}, t)$ of the form*

$$Q(\mathbf{q}, \mathbf{p}, t) = \left(\sum_{i=1}^{n} a_i(\mathbf{q}) p_i\right) - F(\mathbf{q}, t)$$

such that

$$\frac{dQ}{dt} = \{Q, H\} + \frac{\partial Q}{\partial t} = 0\,,$$

Then $\Phi_Q^{(\epsilon)}(L) = L + \epsilon\frac{dF}{dt} + O(\epsilon^2)$, so Q generates a symmetry, whose Noether charge is Q.

Proof. Let me start by proving some simple auxiliary results. Note first that

$$\{q_i, Q\} = \frac{\partial Q}{\partial p_i} = a_i(\mathbf{q}) \tag{6.4.1}$$

which implies, in particular, that

$$\frac{\partial\{q_i, Q\}}{\partial t} = \frac{\partial a_i(\mathbf{q})}{\partial t} = 0\,.$$

Note also that since Q is conserved, and the only explicit time dependence of Q on time is via F, we have

$$\{Q, H\} = \frac{dQ}{dt} - \frac{\partial Q}{\partial t} = -\frac{\partial Q}{\partial t} = \frac{\partial F}{\partial t}\,,$$

so

$$-\{\{Q, H\}, q_i\} = \{q_i, \{Q, H\}\} = \frac{\partial\{Q, H\}}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\frac{\partial F(\mathbf{q}, t)}{\partial t}\right) = 0\,.$$

Using these two results we find that

$$\begin{aligned}
\frac{d\{q_i, Q\}}{dt} &= \frac{\partial\{q_i, Q\}}{\partial t} + \{\{q_i, Q\}, H\} \\
&= \{\{q_i, Q\}, H\} \\
&= -\{\{H, q_i\}, Q\} - \{\{Q, H\}, q_i\} \\
&= \{\{q_i, H\}, Q\}\,.
\end{aligned}$$

where on the third line we have used the Jacobi identity in proposition 6.2.7. Hamilton's equations then imply that

$$\frac{d\{q_i, Q\}}{dt} = \{\dot{q}_i, Q\}\,. \tag{6.4.2}$$

Using these results it is straightforward to compute the change in the Lagrangian due to Q. The Lagrangian in Hamiltonian coordinates is

$$L(\mathbf{q}, \mathbf{p}, t) = \left(\sum_{i=1}^{n} \dot{q}_i(\mathbf{q}, \mathbf{p}) p_i\right) - H(\mathbf{q}, \mathbf{p}, t)\,.$$

Since Q is conserved, we have

$$\begin{aligned}
\{L, Q\} &= \left[\sum_{i=1}^{n} \left(\{\dot{q}_i, Q\} p_i + \dot{q}_i \{p_i, Q\}\right)\right] + \{Q, H\} \\
&= \left[\sum_{i=1}^{n} \left(\{\dot{q}_i, Q\} p_i + \dot{q}_i \{p_i, Q\}\right)\right] - \frac{\partial Q}{\partial t}\,,
\end{aligned}$$

which becomes, using the results above:

$$\begin{aligned}
\{L, Q\} &= \left[\sum_{i=1}^{n} \left(p_i \frac{d}{dt}\{q_i, Q\} + \dot{q}_i \{p_i, Q\}\right)\right] - \frac{\partial Q}{\partial t} \\
&= \left[\sum_{i=1}^{n} \left(\frac{d}{dt}\Big(\{q_i, Q\} p_i\Big) - \dot{p}_i \{q_i, Q\} + \dot{q}_i \{p_i, Q\}\right)\right] - \frac{\partial Q}{\partial t} \\
&= \left[\sum_{i=1}^{n} \left(\frac{d}{dt}\Big(\{q_i, Q\} p_i\Big) - \dot{p}_i \frac{\partial Q}{\partial p_i} - \dot{q}_i \frac{\partial Q}{\partial q_i}\right)\right] - \frac{\partial Q}{\partial t} \\
&= \frac{d}{dt}\left(\sum_{i=1}^{n} \{q_i, Q\} p_i\right) - \frac{dQ}{dt}\,.
\end{aligned}$$

where on the first line we have used (6.4.2), and in going from the third to the fourth the Chain Rule. Finally, using (6.4.1) we find that

$$\sum_{i=1}^{n} \{q_i, Q\} p_i = \sum_{i=1}^{n} a_i p_i = Q + F$$

which implies

$$\{L, Q\} = \frac{d(Q + F - Q)}{dt} = \frac{dF}{dt}\,.$$

Recalling the definition of the Hamiltonian flow operator, this gives

$$\Phi_Q^{(\epsilon)}(L) = L + \epsilon\{L, Q\} + \mathcal{O}(\epsilon^2) = L + \epsilon \frac{dF}{dt} + cO(\epsilon^2)\,.$$

□