§6.6 Monochromatic Waves

🛈 *This section is* ***not examinable.*** 🛈

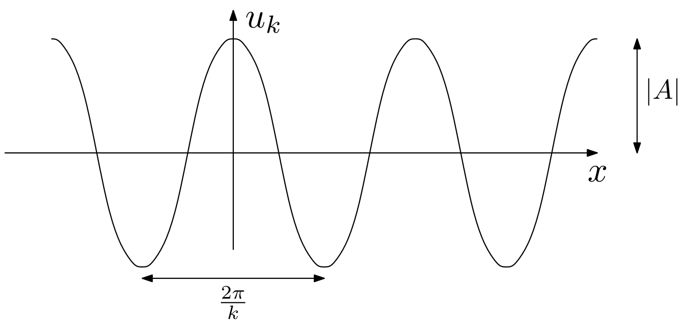
We have already seen that we can write down a general solution to the wave equation, which is solvable as a result of its linearity. Below we will analyse what happens to waves in the presence of boundaries and junctions. This analysis is often simplified if, rather than considering what happens to an arbitrary wave on the string, we 'decompose' the wave into its various constituent wavelengths and consider what happens to each wavelength separately. Using the linearity of the wave equation, the full answer can then be reconstructed by superposing the solution for the constituent wavelengths. A physical analogy would be to imagine the wave to be a light wave. One finds out how red, orange, yellow, green, blue, indigo and violet light behave, and then deduce how a general light wave behaves by mixing the colours together. More mathematically one is simply Fourier decomposing the wave, and using linearity to analyse each Fourier mode separately. For example for a right moving wave we can write as a sum, or more precisely an integral over waves with different frequencies as follows:

$$u(x,t) = f(x-ct) = \int_{-\infty}^{\infty} dk A(k) e^{ik(x-ct)}.$$

The solutions with a definite frequency, or monochromatic waves, are $A(k)e^{ik(x-ct)}$. We have chosen to work with complex exponentials rather than coses and sines, as this makes life easier, but if we need to recover a real solution we can take instead

$$u_k = \Re\left(A(k)e^{ik(x-ct)}\right) = \Re\left(|A|e^{i\theta}e^{ik(x-ct)}\right) = |A|\cos(k(x-ct)+\theta).$$

The graph of u_k shows that $|A|$ is the amplitude of the wave and that the wavelength is $2\pi/k$.



A monochromatic wave moving to the left is given by $u(x,t) = A(k)e^{-ik(x+ct)}$, or we can again take the real part of this to obtain a real solution.

Let us calculate the energy flux of a monochromatic wave. The expression $T_{xt} = -\tau u_t u_x$ we derived measures the flux of energy carried by a solution past a point moving from left to right, so we should expect the answer to be positive for a right moving wave. Taking our solution to be $u(x,t) = u_k$ defined above we see that the flux is given by

$$\begin{aligned} T_{xt} = -\tau (u_k)_t (u_k)_x &= -\tau \left(kc|A|\sin(k(x-ct)+\theta)\right)\left(-k|A|\sin(k(x-ct)+\theta)\right) \\ &= \tau c k^2 |A|^2 \sin^2(k(x-ct)+\theta) \end{aligned}$$

which is clearly positive although it fluctuates with time. If we average over a whole period we see that the average energy passing a point per unit time is given by

$$\frac{kc}{2\pi}\int_0^{\frac{2\pi}{kc}} \tau c k^2 |A|^2 \sin^2(k(x-ct)+\theta)dt = \frac{\tau c k^2 |A|^2}{2}.$$

Note that the energy flux proved to be positive as we had predicted. If we had performed the same calculation on a left moving wave $u = \Re\left(A(k)e^{-ik(x+ct)}\right)$ we would find the average flux to be $-\tau c k^2 |A|^2/2$.