3. ~~We now generalize the result of the previous problem~~. Let us switch to polar coordinates (r, θ), defined by

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

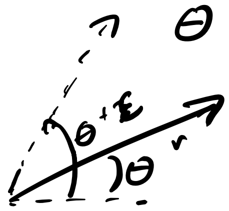
and assume that we have a Lagrangian of the form

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta)$$

for some function $V(r,\theta)$. Find the most general form of $V(r,\theta)$ compatible with arbitrary rotations around $r = 0$ being symmetries.

In polar coordinates a rotation around $r=0$ is:

$$r \to r \qquad \dot{r} \to \dot{r}$$
$$\theta \to \theta + \varepsilon \qquad \dot{\theta} \to \dot{\theta}$$



$$L \to L(r, \theta+\varepsilon, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta+\varepsilon)$$

Taylor expansion in ε

$$\Rightarrow L(r, \theta+\varepsilon, \dot{r}, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta) - \varepsilon\frac{\partial V}{\partial \theta} + \mathcal{O}(\varepsilon^2)$$

$$= L(r, \theta, \dot{r}, \dot{\theta}) - \varepsilon\frac{\partial V}{\partial \theta} + \mathcal{O}(\varepsilon^2)$$

We want to <u>impose</u> that this is of the form

$$L \to L + \varepsilon\frac{dF(r,\theta,t)}{dt} + \mathcal{O}(\varepsilon^2)$$

We need to <u>impose</u> $\frac{\partial V}{\partial \theta} = -\frac{dF}{dt}$ for some F.

Explicitly:

Chain rule

$$\frac{dF(r,\theta,t)}{dt} = \underbrace{\frac{\partial F}{\partial r}}_{\text{function of } r,\theta,t} \cdot \dot{r} + \frac{\partial F}{\partial \theta}\dot{\theta} + \frac{\partial F}{\partial t}$$

$\frac{\partial V(r,\theta)}{\partial \theta}$ is a function of r, θ only (it does not depend on $\dot{r}, \dot{\theta}, t$)

Comparing the terms proportional to velocities:

$$\frac{\partial F}{\partial r} = 0 \quad \text{and} \quad \frac{\partial F}{\partial \theta} = 0$$

$\Rightarrow$ F depends only on t: F(t).

$$\underbrace{\frac{\partial V(r,\theta)}{\partial \theta}}_{\text{depends on } r,\theta} = -\underbrace{\frac{dF(t)}{dt}}_{\text{depends on } t}$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial V(r,\theta)}{\partial \theta} &= k \\ \frac{dF(t)}{dt} &= -k \end{aligned}\right\} \text{ for some constant } k.$$

If we integrate these equations we get:

$$V(r,\theta) = k\theta + P(r)$$ ← arbitrary function of r.

$$F(t) = -kt + d$$ ← constant

We generally impose $\theta \cong \theta + 2\pi$ so we generally additionally require

$$V(r,\theta) = V(r,\theta+2\pi)$$

$$\Rightarrow k = 0$$

$$\Rightarrow \boxed{\begin{aligned} V(r,\theta) &= P(r) \\ F(t) &= d \to 0. \end{aligned}}$$