1. A particle of unit mass moves in a two-dimensional plane parametrised by the Cartesian coordinates (x, y) without friction.

   (a) Find the kinetic energy of the system in terms of the polar coordinates (r, θ), defined via $x = r\cos(\theta)$, $y = r\sin(\theta)$.

$$T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

$$\left.\begin{aligned} \dot{x} &= \dot{r}\cos(\theta) + r(-\sin\theta)\dot{\theta} \\ \dot{y} &= \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta} \end{aligned}\right\}$$

$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

(b) Assuming that there is a rotationally symmetric potential $V(r) = \frac{r^6}{6}$, write the Lagrangian for the system in polar coordinates, and derive the Euler-Lagrange equations of motion associated to these polar coordinates. (You do not need to solve them.)

$$L = T - V$$
$$= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{r^6}{6}$$

Euler-Lagrange equations:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dt}\left(\dot{r}\right) - \left(r\dot{\theta}^2 - r^5\right)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(r^2\dot{\theta}\right) - 0$$

(c) Which coordinate is ignorable? Write the associated generalised momentum J, and show that it is conserved.

θ is ignorable because $\frac{\partial L}{\partial \theta} = 0$.

The associated generalised momentum:

$$\frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta}$$

It is conserved due to the θ Euler-Lagrange equation.

(d) Find an equation of motion for r only (without θ or $\dot{\theta}$ appearing anywhere) in terms of the conserved charge J.

$$\ddot{r} + r^5 - r\dot{\theta}^2 = 0 \qquad (\text{"}r\text{" E-L equation})$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \Rightarrow J = r^2\dot{\theta} \text{ is conserved}$$

$$\Rightarrow \dot{\theta} = J/r^2$$

$$\ddot{r} + r^5 - r\left(J/r^2\right)^2 = 0$$

$$\Updownarrow$$

$$\ddot{r} + r^5 - J^2/r^3 = 0$$