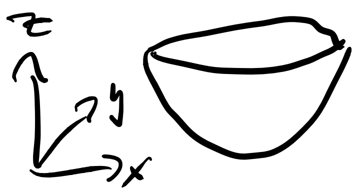
1. A particle of unit mass is restricted to move on the surface of a bowl parametrized, in Cartesian coordinates for $\mathbb{R}^3$, by the equation $z = x^2 + y^2$. The gravitational potential acting on the particle is $V = gz$.

   (a) Choose q_1, q_2 as generalised coordinates, with

$$x = q_1 \cos(q_2)$$
$$y = q_1 \sin(q_2)$$
$$z = q_1^2$$

   Express the kinetic energy of the particle in the (q_1, q_2) coordinates.



$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

$$\dot{x} = \dot{q}_1 \cos(q_2) + q_1 \left(-\sin(q_2) \dot{q}_2 \right)$$
$$\dot{y} = \dot{q}_1 \sin(q_2) + q_1 \cos(q_2) \dot{q}_2$$

$$\dot{z} = 2 q_1 \dot{q}_1$$

$$T = \frac{1}{2} m \Big[\left(\dot{q}_1 \cos(q_2) - q_1 \sin(q_2) \dot{q}_2 \right)^2$$
$$+ \left(\dot{q}_1 \sin(q_2) + q_1 \cos(q_2) \dot{q}_2 \right)^2$$
$$+ 4 q_1^2 \dot{q}_1^2 \Big]$$

$$= \frac{1}{2} m \Big[\dot{q}_1^2 \overbrace{\left(\cos^2(q_2) + \sin^2(q_2) \right)}^{=1}$$
$$+ q_1^2 \dot{q}_2^2 + 4 q_1^2 \dot{q}_1^2 \Big]$$

(b) Write the Lagrangian and the Euler-Lagrange equations of motion in the (q_1, q_2) coordinates. (You do not need to solve the resulting equations.)

$$L = T - V = \frac{1}{2} m \left[\dot{q}_1^2 + q_1^2 \dot{q}_2^2 + 4 q_1^2 \dot{q}_1^2 \right] - g q_1^2$$

Euler-Lagrange eq's:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \frac{d}{dt}\left(m \dot{q}_1 + m 4 q_1^2 \dot{q}_1\right)$$

$$- \left(m q_1 \dot{q}_2^2 + 4 m q_1 \dot{q}_1^2 - 2 g q_1\right)$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \frac{d}{dt}\left(m q_1^2 \dot{q}_2\right) - 0$$

(c) Which coordinate is ignorable? Write the associated generalised momentum P, and show that it is conserved.

q_2 is ignorable because it does not appear in L.

$$p := \frac{\partial L}{\partial \dot{q}_2} = m q_1^2 \dot{q}_2$$

it's conserved due to the q_2 E-L equation of motion.

(d) Show that the Euler-Lagrange equations of motion do not change if we replace the Lagrangian L by $L' = L + f(t)$, for any arbitrary function $f(t)$ of time, and an arbitrary Lagrangian L.

$$\frac{d}{dt}\left(\frac{\partial(L+f)}{\partial \dot{q}_i}\right) - \frac{\partial(L+f)}{\partial q_i} = 0$$

$\Downarrow$ By linearity of derivatives:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \underbrace{\left(\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) - \frac{\partial f}{\partial q_i}\right)}_{=0} = 0$$

f is a function of t only

$$\Rightarrow \frac{\partial f}{\partial \dot{q}_i} = \frac{\partial f}{\partial q_i} = 0$$