2. A system is described by a Lagrangian $L = L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n)$ which does not explicitly depend on time.

   (a) Use the Euler-Lagrange equations to show that

$$E \equiv \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L$$

   is conserved.

$$\frac{dE}{dt} = \left(\sum_{i=1}^{n} \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) \right) - \frac{dL}{dt}$$

$$\frac{dL}{dt} = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right)$$

chain rule

$$= \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$$

$$\frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) = \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right)$$

$$= \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial L}{\partial q_i}$$

Euler-Lagrange

$$\Rightarrow \frac{dE}{dt} = 0$$

(b) Find E for the Lagrangian

$$L_1 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + (\dot{x} - \dot{y})\, z\,.$$

$$E = \left(\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + z$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} - z$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}} - L$$

$$= \dot{x}(m\dot{x} + z) + \dot{y}(m\dot{y} - z) + m\dot{z}^2 -$$
$$- \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - (\dot{x} - \dot{y})z$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

(c) Which co-ordinates are ignorable for the Lagrangian L_1 and what are the corresponding conserved quantities? Use these to show that $\dot{x} + \dot{y}$ is conserved.

x and y are ignorable.

$\Rightarrow p_x := \frac{\partial L}{\partial \dot{x}}$ and $p_y := \frac{\partial L}{\partial \dot{y}}$ are conserved.

$$\frac{d}{dt} p_x \overset{\text{Euler-Lagrange}}{=} \frac{\partial L}{\partial x} \underset{x \text{ is ignorable}}{=} 0$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + z$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} - z$$

$$\frac{dp_x}{dt} = 0 \text{ and } \frac{dp_y}{dt} = 0 \Rightarrow \frac{d}{dt}(p_x + p_y) = 0$$

$$p_x + p_y = m(\dot{x} + \dot{y})$$

$$\overset{(m \neq 0)}{\Longrightarrow} \frac{d}{dt}(\dot{x} + \dot{y}) = 0$$

(d) Find the Euler-Lagrange equations for L_1.

Euler-Lagrange equation for x:

$$\frac{d}{dt} p_x = 0 \quad \left[\Leftrightarrow \frac{d}{dt}(m\dot{x} + z) = 0 \right]$$

EL eq'n for y:

$$\frac{d}{dt} p_y = 0 \Leftrightarrow \frac{d}{dt}(m\dot{y} - z) = 0$$

EL for z:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}$$

$$\frac{\partial L}{\partial z} = \dot{x} - \dot{y}$$

$$m\ddot{z} = \dot{x} - \dot{y}$$

(e) Compare the Euler-Lagrange equations for L_1 with those for

$$L_2 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \dot{z}(x - y)\,.$$

Exercise

If you compute the EL equations you get <u>the same</u> equations.

$$L_1 = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + (\dot{x} - \dot{y})z$$

$$\begin{aligned} L_1 - L_2 &= (\dot{x} - \dot{y})z + \dot{z}(x-y) \\ &= \frac{d}{dt}\left(z(x-y)\right) \\ &= \frac{d}{dt}F(x,y,z) \end{aligned}$$