8. Assume that we have a system with Lagrangian

$$L = \frac{1}{2}(aq_1 - q_2)^2(\dot{q}_1^2 + \dot{q}_2^2) - \cos(aq_1 - q_2)$$

for some constant a.

(a) Consider the transformation

$$q_1 \to q_1' = q_1 + \epsilon\,,$$
$$q_2 \to q_2' = q_2 + \epsilon\,.$$

For which value of a is this transformation a symmetry?

$\dot{q}_1 \to \dot{q}_1$ and $\dot{q}_2 \to \dot{q}_2$

$$L \to L' = \frac{1}{2}\left(a(q_1+\varepsilon) - (q_2+\varepsilon)\right)^2(\dot{q}_1^2 + \dot{q}_2^2)$$
$$- \cos(a(q_1+\varepsilon) - q_2 + \varepsilon)$$
$$= \frac{1}{2}(\underbrace{aq_1 - q_2 + (a-1)\varepsilon}_{})^2(\dot{q}_1^2 + \dot{q}_2^2)$$
$$- \cos(aq_1 - q_2 + (a-1)\varepsilon)$$

To first order in ε:

$$(aq_1 - q_2 + (a-1)\varepsilon)^2 \simeq (aq_1 - q_2)^2 + 2\varepsilon(a-1)(aq_1 - q_2) + \mathcal{O}(\varepsilon^2)$$

$$\cos(aq_1 - q_2 + (a-1)\varepsilon)$$
$$= \cos(aq_1 - q_2)\cos((a-1)\varepsilon)$$
$$- \sin(aq_1 - q_2)\sin((a-1)\varepsilon)$$

$(\cos(\varepsilon) \simeq 1 + \mathcal{O}(\varepsilon^2)\,,\ \sin(\varepsilon) \simeq \varepsilon + \mathcal{O}(\varepsilon^3))$

$$= \cos(aq_1 - q_2) - (a-1)\varepsilon\sin(aq_1 - q_2)$$

$$+\mathcal{O}(\varepsilon^2)$$

Collecting terms:

$$L' = L + (aq_1 - q_2)((a-1)\varepsilon)(\dot{q}_1^2 + \dot{q}_2^2) + \sin(aq_1 - q_2)(a-1)\varepsilon + \mathcal{O}(\varepsilon^2)$$

This is a symmetry if

$$L' = L + \varepsilon \frac{dF(q_1, q_2, t)}{dt} + \mathcal{O}(\varepsilon^2)$$

This is only possible if $a=1$.

To see this, assume that F exists. Then:

$$\frac{dF(q_1,q_2,t)}{dt} = \frac{\partial F}{\partial q_1}\dot{q}_1 + \frac{\partial F}{\partial q_2}\dot{q}_2 + \frac{\partial F}{\partial t}$$

so, comparing terms multiplying $\dot{q}_1$, $\dot{q}_2$:

$$\frac{\partial F(q_1,q_2,t)}{\partial q_1} = (aq_1 - q_2)(a-1)\dot{q}_1$$

$$\frac{\partial F(q_1,q_2,t)}{\partial q_2} = (aq_1 - q_2)(a-1)\dot{q}_2$$

Since the left hand side does not depend on $\dot{q}_1$, $\dot{q}_2$, the only solution is $a=1$.