8. (a) State d'Alembert's solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(Note: $c^2 = 1$)

where $-\infty < x < \infty$, and use it to solve that equation subject to the initial condition $u(x,0) = e^{-x^2}$, $\partial u(x,0)/\partial t = 0$.

D'Alembert's solution:

$$u(x,t) = f(x-ct) + g(x+ct)$$

$$\overset{c=1}{=} f(x-t) + g(x+t)$$

From the initial conditions:

$$e^{-x^2} = u(x,0) = f(x) + g(x)$$

By the chain rule: $\frac{\partial(x-t)}{\partial t} = -1$

$$0 = u_t(x,0) = -f'(x-0) + g'(x)$$

$$= -f'(x) + g'(x)$$

$$\Rightarrow f(x) = g(x) + \underbrace{\text{constant}}_{\text{"}b\text{"}}$$

$$= g(x) + b$$

$$\Rightarrow 2g(x) + b = e^{-x^2} \Rightarrow g(x) = \frac{1}{2}(e^{-x^2} - b)$$

$$f(x) = \frac{1}{2}(e^{-x^2} + b)$$

$$\Rightarrow u(x,t) = \frac{1}{2}\left(e^{-(x-t)^2} - b\right) + \frac{1}{2}\left(e^{-(x+t)^2} + b\right)$$

$$= \frac{1}{2}\left(e^{-(x-t)^2} + e^{-(x+t)^2}\right)$$

(b) Define

$$X_n(a,b) = \int_a^b \left(\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} \right)^n + \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \right)^n \right) dx$$

where n is a positive integer. Use the wave equation to show that $\rightarrow u_{tt} = u_{xx}$

$$\frac{d}{dt} X_n(a,b) = \left[g_n \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \right) \right]_a^b$$

for some function g_n which you should determine. Find suitable conditions on $u(x,t)$ at $x = \pm\infty$, for $X_n(-\infty, \infty)$ to be conserved.

$$\frac{d}{dt} X_n(a,b) = \int_a^b \frac{\partial}{\partial t} \left((u_x + u_t)^n + (u_x - u_t)^n \right) dx$$

chain rule

$$= \int_a^b dx \left[n (u_x + u_t)^{n-1} (u_{xt} + u_{tt}) + n (u_x - u_t)^{n-1} (u_{xt} - u_{tt}) \right]$$

$$= \int_a^b \Big[\underbrace{n (u_x + u_t)^{n-1} (u_{xt} + \underbrace{u_{xx}}_{\partial_x (u_t + u_x)})}_{\partial_x (u_x + u_t)^n} + \underbrace{n (u_x - u_t)^{n-1} \underbrace{(u_{xt} - u_{xx})}_{\partial_x (u_t - u_x)}}_{-\partial_x (u_x - u_t)^n} \Big] dx$$

$$= \int_a^b \partial_x \left((u_x + u_t)^n - (u_x - u_t)^n \right) dx$$

$$= \left[(u_x + u_t)^n - (u_x - u_t)^n \right]_a^b$$

$$\Rightarrow g_n = (u_x + u_t)^n - (u_x - u_t)^n$$

One possibility for X_n to be conserved is

$$\lim_{x \to \pm\infty} u_x = \lim_{x \to \pm\infty} u_t = 0$$

(c) Show that when X_n is evaluated for d'Alembert's general solution to the wave equation it becomes a sum of two contributions, one for a right-moving wave and one for a left-moving wave.

$$u_x = \frac{\partial}{\partial x}\left(f(x-t) + g(x+t)\right) \longrightarrow f(h(x))$$

$$h(x) = x - t$$

$$= f'(x-t) + g'(x+t) \qquad \frac{\partial f}{\partial x} = \frac{df}{dh}\cdot\frac{\partial h}{\partial x} = f'\cdot 1 = f'$$

$$u_t = \frac{\partial}{\partial t}\left(f(x-t) + g(x+t)\right)$$

$$= -f'(x-t) + g'(x+t)$$

$$\Rightarrow \quad u_x + u_t = 2g'(x+t)$$

$$u_x - u_t = 2f'(x-t)$$

$$X_n = \int_a^b \left[(u_x + u_t)^n + (u_x - u_t)^n\right] dx$$

$$= \int_a^b \left[(2g'(x+t))^n + (2f'(x-t))^n\right] dx$$

$$f'(\xi) = \frac{d}{d\xi}\left(\frac{1}{2} e\right.$$

(d) Evaluate $X_2(-\infty, \infty)$ for the initial conditions given in part (a). You may find the result $\int_{-\infty}^{\infty} x^2\, e^{-a\,x^2}\, dx = \sqrt{\pi/a^3}/2$ useful.

$$X_2(-\infty,\infty) = \int_{-\infty}^{\infty} \left((u_x + u_t)^2 + (u_x - u_t)^2 \right) dx$$

$$u_x + u_t = 2g'(x+t)$$

$$u_x - u_t = 2f'(x-t)$$

The initial conditions had

$$f(\xi) = \frac{1}{2}(e^{-\xi^2} + S) \Rightarrow f' = -\xi\, e^{-\xi^2}$$

$$g(\xi) = \frac{1}{2}(e^{-\xi^2} - S) \Rightarrow g' = -\xi\, e^{-\xi^2}$$

$$\Rightarrow X_2 = \int_{-\infty}^{\infty} dx \left[\underbrace{\left(2(x+t) e^{-\frac{1}{2}(x+t)^2}\right)^2}_{\text{From } g'} + \left(2(x-t) e^{-\frac{1}{2}(x-t)^2}\right)^2 \right]$$

<u>Note</u> : X_2 is conserved

$\Rightarrow$ we can evaluate at $t=0$

$$\Rightarrow X_2 = \int_{-\infty}^{\infty} 8\, x^2\, e^{-x^2} = 4\sqrt{\pi}$$