3. The Lagrangian of a system with generalised co-ordinates x and y is

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 - 5x^2 - 4xy - 2y^2\right)$$

(a) Find the Euler-Lagrange equations for x and y.

$$\overset{\overset{0}{\shortparallel}}{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}} = \ddot{x} - \left(\frac{1}{2}\frac{\partial}{\partial x}\left(-5x^2 - 4xy\right)\right)$$

$$= \ddot{x} + 5x + 2y$$

$$\overset{\overset{0}{\shortparallel}}{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y}} = \ddot{y} - \frac{1}{2}\frac{\partial}{\partial y}\left(-2y^2 - 4xy\right)$$

$$= \ddot{y} + 2y + 2x$$

(b) Write these Euler-Lagrange equations in the matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} = -\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix}$$

and find the two by two matrix $\mathbf{A}$.

$$\frac{d^2}{dt^2}\begin{pmatrix} x \\ y \end{pmatrix} + A\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$A \rightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{pmatrix} A_{11}x + A_{12}y \\ A_{21}x + A_{22}y \end{pmatrix} = 0$$

(c) Find the eigenvalues and the eigenvectors of the matrix **A**.

$$\det(\lambda - A) = 0$$

$$\det\left(\lambda - \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\right) = \det\begin{pmatrix} \lambda-5 & -2 \\ -2 & \lambda-2 \end{pmatrix}$$

$$= (\lambda-5)(\lambda-2) - 4 = \lambda^2 - 7\lambda + 6$$

$$\lambda = \frac{+7 \pm \sqrt{49-24}}{2} = \frac{+7 \pm \sqrt{25}}{2}$$

$$= \frac{7 \pm 5}{2} \Rightarrow \begin{array}{l} \lambda^{(1)} = 1 \\ \lambda^{(2)} = 6 \end{array}$$

$$A\vec{v} = \lambda \vec{v}$$

$$A\vec{v}^{(1)} = \lambda^{(1)}\vec{v}^{(1)} \Rightarrow \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\vec{v}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\vec{v}^{(1)}$$

$$\Leftrightarrow \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\vec{v}^{(1)} = 0 \Rightarrow 2v_1^{(1)} + v_2^{(1)} = 0$$

$$\Rightarrow \vec{v}^{(1)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \left(\text{Check: } \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \checkmark \right)$$

Similarly for $\lambda^{(2)} = 6$:

$$\vec{v}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(d) Use these to construct the general solution of the Euler-Lagrange equation for the real column vector

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \vec{v}^{(1)} \left(\alpha^{(1)} \cos\left(\sqrt{\lambda^{(1)}}\, t\right) + \beta^{(1)} \sin\left(\sqrt{\lambda^{(1)}}\, t\right)\right) + \vec{v}^{(2)} \left(\alpha^{(2)} \cos\left(\sqrt{\lambda^{(2)}}\, t\right) + \beta^{(2)} \sin\left(\sqrt{\lambda^{(2)}}\, t\right)\right)$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \left(\alpha^{(1)} \cos(t) + \beta^{(1)} \sin(t)\right) + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \left(\alpha^{(2)} \cos(\sqrt{6}\, t) + \beta^{(2)} \sin(\sqrt{6}\, t)\right)$$

(e) Initially the values of x and y and its derivative are $(x, y) = (2, 1)$ and $(\dot{x}, \dot{y}) = (0, 0)$. Find their subsequent values.

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} = \Big[\begin{pmatrix} 1 \\ -2 \end{pmatrix} \left[\alpha^{(1)} (-\sin(t)) + \beta^{(1)} \cos(t) \right] + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \left[\alpha^{(2)} (-\sqrt{6} \sin(\sqrt{6}\, t)) + \beta^{(2)} (\sqrt{6} \cos(\sqrt{6}\, t)) \right] \Big]_{t=0}$$

$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix} \beta^{(1)} + \sqrt{6}\, \beta^{(2)} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \beta^{(1)} = \beta^{(2)} = 0 \quad \text{(by linear independence)}$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \alpha^{(1)} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \alpha^{(2)}$$

Take the dot product with $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 + 4 = 5$$

$$0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}}^{5} \alpha^{(1)} + \overbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}}^{=0} \alpha^{(2)}$$

$$\Rightarrow 5 \alpha^{(1)} = 0 \Rightarrow \alpha^{(1)} = 0$$

Taking dot product with $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5$$

$$5 = 0 \cdot \alpha^{(1)} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \alpha^{(2)}$$

$$\Rightarrow \quad \alpha^{(\ldots)} = 1$$

$$\Rightarrow \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos(\sqrt{6}\, t)$$