8. The displacement $u(x, t)$ of a particular string satisfies the wave-equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

and the energy of the portion of the string between $x = a$ and $x = b$ is given by

$$E(a, b) = \frac{1}{2} \int_a^b \left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right) dx.$$

(a) Use the wave-equation to show that

$$\frac{1}{2} \frac{\partial}{\partial t} \left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right) = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \frac{\partial u}{\partial x} \right)$$

subject to the second partial derivatives of u being continuous.

$$\frac{1}{2} \frac{\partial}{\partial t} \left(u_t^2 + u_x^2 \right) = u_t u_{tt} + u_x u_{tx} \qquad \text{(LHS)}$$

$$\frac{\partial}{\partial x} \left(u_t u_x \right) = u_{tx} u_x + u_t u_{xx} \qquad \text{(RHS)}$$

EOM: $u_{tt} = u_{xx}$

LHS $\rightarrow$ $u_t u_{xx} + u_x u_{tx} = \text{(RHS)}$

(b) Hence show that

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b.$$

$$\frac{d}{dt}E(a,b) = \frac{d}{dt}\int_a^b \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right) dx$$

$$= \int_a^b \frac{\partial}{\partial t}\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right) dx$$

$$= \int_a^b \frac{\partial}{\partial x}\left(u_t u_x\right) dx$$

$$= \left[u_t u_x\right]_a^b$$

(c) Define r and s by

$$\overset{u_t + u_x}{r = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}}, \quad \overset{u_t - u_x}{s = \frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}}.$$

Show that for any differentiable function H

$$\frac{\partial H(r)}{\partial t} = \frac{\partial H(r)}{\partial x}, \quad \frac{\partial H(s)}{\partial t} = -\frac{\partial H(s)}{\partial x},$$

and also show that this implies

$$\frac{d}{dt}\int_a^b H(r)\,dx = [H(r)]_{x=a}^{x=b}, \quad \frac{d}{dt}\int_a^b H(s)\,dx = -[H(s)]_{x=a}^{x=b}.$$

$$\frac{\partial H(r)}{\partial t} = H'(r)\frac{\partial r}{\partial t} = H'(r)(u_{tt} + u_{tx})$$

$$\frac{\partial H(r)}{\partial x} = H'(r)\frac{\partial r}{\partial x} = H'(r)(u_{tx} + u_{xx})$$

$$\underset{u_{tt} = u_{xx}}{=} H'(r)(u_{tx} + u_{tt}) = \frac{\partial H}{\partial t}$$

$$\frac{\partial H(s)}{\partial t} = H'(s)\frac{\partial s}{\partial t} = H'(s)(u_{tt} - u_{tx})$$

$$\frac{\partial H(s)}{\partial x} = H'(s)\frac{\partial s}{\partial x} = H'(s)(u_{tx} - u_{xx})$$

$$\underset{u_{tt} = u_{xx}}{=} -\frac{\partial H(s)}{\partial t}$$

$$\frac{d}{dt}\int_a^b H(r)\,dx = \int_a^b \frac{\partial H}{\partial t}\,dx \overset{\frac{\partial H(r)}{\partial t} = \frac{\partial H(r)}{\partial x}}{=} \int_a^b \frac{\partial H(r)}{\partial x}\,dx$$

$$= [H(r)]_a^b$$

(d) The string is physically altered so that the wave equation is modified to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \sin u$$

and as a consequence the energy is modified to

$$E(a,b) = \frac{1}{2}\int_a^b \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + V(u) \right) dx.$$

for some function $V(u)$. Find this function given that the energy still satisfies

$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b.$$

$$\mathcal{E} = \frac{1}{2}\left(u_t^2 + u_x^2 + V(u)\right)$$

$$\Downarrow \qquad \mathcal{E} = T_{tt} = T_{00}$$

$$\mathcal{L} = \frac{1}{2}\left(u_t^2 - u_x^2 - V(u)\right)$$

$$\Downarrow$$

$$0 = \frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right)$$

$$= -\frac{1}{2}V'(u) + u_{xx} - u_{tt} \Leftrightarrow u_{tt} = u_{xx} - \frac{1}{2}V'(u)$$

$$\Rightarrow -\frac{1}{2}V'(u) = \sin(u) \Rightarrow V(u) = 2\cos(u)$$

$$+ \text{constant}$$

(e) Solutions to the modified wave equation can be found in the form $u = f(x-ct)$ provided f satisfies a second order ordinary differential equation. Find this equation and show that it implies that

$$\frac{1}{2}f'(x)^2 + \frac{\cos f(x)}{c^2-1}$$

is independent of x.

$u_{tt} = u_{xx} + \sin(u)$

For $u(x,t) = f(x-ct)$

$u_t = -c\, f'(x-ct)$

$u_{tt} = c^2 f''(x-ct)$

$u_x = f'(x-ct)$

$u_{xx} = f''(x-ct)$

$c^2 f''(x-ct) = f''(x-ct) + \sin(f(x-ct))$

$$\boxed{f'' = \frac{\sin(f)}{c^2-1}}$$

$$\frac{d}{dx}\left(\frac{1}{2}(f')^2 + \frac{\cos(f)}{c^2-1}\right)$$

$$= f' f'' - \frac{\sin(f)}{c^2-1} f'$$

$$= f' \underbrace{\left(f'' - \frac{\sin(f)}{c^2-1}\right)}_{=0} = 0$$