7. We describe small oscillations of an infinite one-dimensional string by a one-dimensional field $u(x,t)$ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 \quad \Rightarrow \rho = \tau = 1$$

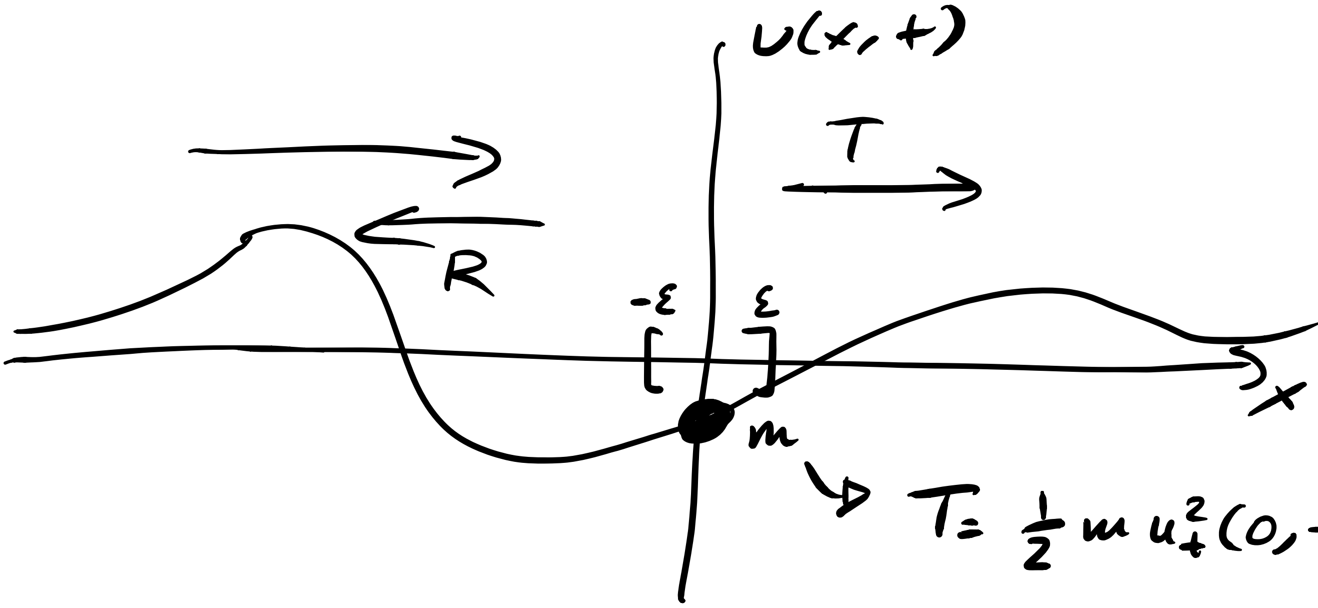
$$\Downarrow$$

where $u_x \equiv \partial u/\partial x$ and $u_t = \partial u/\partial t$. $\qquad c^2 = 1$

(e) Assume that we attach a mass m to the string, fixed at $x = 0$, so that there is an extra finite contribution to the kinetic energy of the string coming from the mass at $x = 0$. (As above, we ignore the effect of gravity.) Taking a monochromatic wave ansatz of the form

$$u(x,t) = \begin{cases} \Re\left((e^{i\omega x} + Re^{-i\omega x})e^{-i\omega t}\right) & \text{for } x < 0 \\ \Re\left(Te^{i\omega x}e^{-i\omega t}\right) & \text{for } x > 0 \end{cases}$$

solve for R and T in the presence of the mass. Interpret what happens for $m \to 0$ and $m \to \infty$.



$\rightarrow T = \frac{1}{2} m u_t^2(0,t)$

- Junction conditions
  1. Continuity
  2. Energy conservation across the junction

① <u>Continuity</u>

$$\lim u(x,t) = \lim u(x,t)$$

$$\lim_{x\to 0^-} \qquad\qquad \lim_{x\to 0^+}$$

$$\mathrm{Re}\left((1+R)e^{-i\omega t}\right) = \mathrm{Re}\left(T e^{-i\omega t}\right)$$

for all t

Choosing $t=0 \Rightarrow \mathrm{Re}(1+R) = \mathrm{Re}(T)$

$t = -\frac{\pi}{2\omega} \Rightarrow \mathrm{Im}(1+R) = \mathrm{Im}(T)$

(or $t = \left(-\frac{\pi}{2} + 2\pi\right)\frac{1}{\omega}$

$\Rightarrow 1+R = T$

Note: $1+R=T$ implies

$$\lim_{x\to 0^-} u_t(x,t) = \lim_{x\to 0^+} u_t(x,t)$$

similarly for u_{tt}. So $u_t(0,t)$ and $u_{tt}(0,t)$ are well defined.

(2) Energy conservation

$$\frac{d}{dt}\left(\lim_{\varepsilon\to 0} E(-\varepsilon,\varepsilon)\right) = \overbrace{(-u_x u_t)\Big|_{x=-\varepsilon}}^{\text{Energy flux from the left}} - \underbrace{(-u_x u_t)\Big|_{x=+\varepsilon}}_{\text{energy flux leaving from right}}$$

$$\lim_{\varepsilon\to 0} E(-\varepsilon,\varepsilon) = \underbrace{\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right)dx}_{=0} + \frac{1}{2} m\, u_t^2(0,t)$$

$$= \frac{1}{2} m\, u_t^2(0,t) + \cancel{\int_0 f(x)\,dx}$$

$$\frac{d}{dt} \lim_{\varepsilon \to 0} E(-\varepsilon, \varepsilon) = \frac{d}{dt}\left(\frac{1}{2} m\, u_t^2(0,t) \right)$$

$$= m\, u_t(0,t)\, u_{tt}(0,t)$$

$$\Rightarrow \quad m\, u_t(0,t)\, u_{tt}(0,t) = \lim_{\varepsilon \to 0^+} u_x(\varepsilon,t)\, u_t(\varepsilon,t) - \lim_{\varepsilon \to 0^-} u_x(\varepsilon,t)\, u_t(\varepsilon,t)$$

Divide by $u_t(0,t)$:

$$\underbrace{m\, u_{tt}(0,t)}_{m\,\mathrm{Re}(T(-i\omega)^2 e^{-i\omega t})} = \overbrace{\lim_{\varepsilon \to 0^+} u_x(\varepsilon,t)}^{\mathrm{Re}(T(i\omega)e^{-i\omega t})} - \underbrace{\lim_{\varepsilon \to 0^-} u_x(\varepsilon,t)}_{\mathrm{Re}((i\omega - Ri\omega)e^{-i\omega t})}$$

$$-\mathrm{Re}(mT\omega^2 e^{-i\omega t}) = \mathrm{Re}(i\omega(T+R-1)e^{-i\omega t})$$

$\Downarrow$ holds for all t

$$-\omega^2 T m = i\omega(T+R-1)$$

$\Downarrow$ together with $T = 1+R$

$$T = \frac{2i}{2i+m} \qquad R = \frac{-m}{2i+m}$$

For $m = 0$: (No mass)

$$T = \frac{2i}{2i} = 1 \qquad R = 0$$

For $m \to \infty$: (Attaching the string at the origin
Dirichlet boundary conditions)

$$T \to 0 \qquad R \to -1$$