Hamiltonian Mechanics - Hamilton's Equations and Poisson Brackets

1. The Poisson bracket of two dynamical variables $A(q_i, p_j),\ B(q_i, p_j)$ is defined by

$$\{A, B\} = \Sigma_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right)$$

where the sum goes over all degrees of freedom i.

   (a) Check that Poisson brackets anticommute: $\{A, B\} = -\{B, A\}$

   (b) Check that they are linear: for numbers a, b, $\{aA+bC, B\} = a\{A, B\}+b\{C, B\}$.

   (c) Show that they satisfy the Leibniz rule

$$\{A, BC\} = \{A, B\}C + B\{A, C\}$$

2. Show that the coordinates and momenta have the following "canonical" Poisson brackets.

$$\{q_i, q_j\} = 0$$
$$\{p_i, p_j\} = 0$$
$$\{q_i, p_j\} = \delta_{ij}$$

3. Show, by using Hamilton's equations of motion

$$\dot{q}_i = \{q_i, H(\mathbf{p}, \mathbf{q}, t)\} = \frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial p_i} \quad ; \quad \dot{p}_i = \{p_i, H(\mathbf{p}, \mathbf{q}, t)\} = -\frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial q_i}$$

that

$$\frac{dH(\mathbf{p}, \mathbf{q}, t)}{dt} = \frac{\partial H(\mathbf{p}, \mathbf{q}, t)}{\partial t}$$

4. If A_1, A_2 are functions of p_i, q_i that do not depend explicitly on t, and are conserved:

$$\{A_1, H\} = 0$$
$$\{A_2, H\} = 0$$

show that their Poisson bracket $A_3(p, q) = \{A_1, A_2\}$ is also conserved:

$$\{A_3, H\} = 0\,.$$

Show that the same is true even if A_1 and A_2 depend explicitly on time. That is, assuming that

$$\{A_1, H\} + \frac{\partial A_1}{\partial t} = 0$$

and similarly for A_2

$$\{A_2, H\} + \frac{\partial A_2}{\partial t} = 0$$

prove that $A_3(p, q, t) = \{A_1, A_2\}$ is conserved:

$$\{A_3, H\} + \frac{\partial A_3}{\partial t} = 0\,.$$

Hint 1: *You might want to remind yourself of the Jacobi identity for Poisson brackets, shown in problem* (7) *below.*

Hint 2: *You might want to start by showing*

$$\frac{\partial}{\partial t}\{A, B\} = \left\{\frac{\partial A}{\partial t}, B\right\} + \left\{A, \frac{\partial B}{\partial t}\right\}$$

for arbitrary functions $A(p, q, t)$*,* $B(p, q, t)$*.*

5. A relativistic particle has a Lagrangian

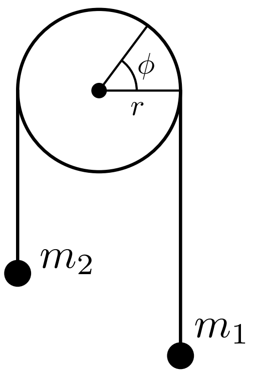
$$L = -mc\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}.$$

Find the corresponding Hamiltonian

$$H = \left(\sum_{i=1}^{n} p_i \dot{q}_i\right) - L$$

for the particle, in terms of the momenta p_x, p_y and p_z. Show that if we define a four vector $p^\mu = (H/c, p_x, p_y, p_z)$, the quantity $(H/c)^2 - (p_x)^2 - (p_y)^2 - (p_z)^2$ made from its components is a constant you should determine.

6. A pair of weights of mass m_1 and m_2 respectively are attached by an inextendable string which passes over a smooth light pulley of radius r which turns through an angle ϕ, as in the diagram below.



Show that the Lagrangian for this system can be written as

$$L = \frac{1}{2}(m_1 + m_2)r^2\dot{\phi}^2 - gr(m_1 - m_2)\phi$$

Find the Hamiltonian for this system and write down Hamilton's equations of motion. Show that

$$r\ddot{\phi} = \frac{(m_2 - m_1)g}{(m_1 + m_2)}\,.$$

7. Show that the Poisson bracket satisfies the Jacobi identity

$$\{\{A, B\}, C\} + \{\{C, A\}, B\} + \{\{B, C\}, A\} = 0.$$

by expanding everything out and looking for cancellations.

8. A charged particle of unit mass moves in two dimensions under the influence of a dipole. The Lagrangian for this motion is

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \mu\frac{\cos\theta}{r^2}$$

where r, θ are the plane polar coordinates of the particle, and μ is the constant strength of the dipole. Derive the generalised momenta p_r and p_θ conjugate to r and θ. Write down the Hamiltonian and obtain Hamilton's equations of motion. Show that

$$p_\theta^2 + 2\mu\cos\theta = \alpha \qquad \text{and} \qquad \dot{r}^2 + \frac{\alpha}{r^2} = 2E$$

where α is a constant and E is the total energy.

9. The dynamics of a system of N degrees of freedom is specified by a Lagrangian $L(q, \dot{q}, t)$, where $q = (q_1, q_2, ..., q_n)$. We have shown before that for any function $f(q, t)$

$$L' = L + \frac{df}{dt} = L + \dot{q}_k\frac{\partial f}{\partial q_k} + \frac{\partial f}{\partial t}$$

produces the same Lagrangian equations of motion.

   (a) Find the canonical momenta and Hamiltonian for the new Lagrangian. How do they relate to those for the old Lagrangian?

   (b) Show that the Hamiltonians corresponding to L and L' produce equivalent equations of motion.

10. Consider a system depending on a single degree of freedom q with conjugate momentum p, and consider the change of variables

$$Q = q\cos(\alpha) + p\sin(\alpha)$$
$$P = -q\sin(\alpha) + p\cos(\alpha)$$

for some constant but arbitrary α, which mixes coordinates with momenta.

(a) Express the Hamiltonian in terms of the new variables P, Q in the particular case of the harmonic oscillator

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$$

for the case $\alpha = \frac{\pi}{2}$. Show that, when expressed in these variables, the equations of motion arising from this new Hamiltonian

$$\dot{Q} = \frac{\partial H(P,Q)}{\partial P}$$
$$\dot{P} = -\frac{\partial H(P,Q)}{\partial Q}$$

are equivalent to the original ones.

(b) More generally, for any arbitrary Hamiltonian $H(p, q)$, and any α, show that the equations of motion are still of Hamilton's form in the new variables.

(c) For the original coordinates we have that

$$\{p, p\} = \{q, q\} = 0 \qquad ; \qquad \{q, p\} = 1\,.$$

(These are known as the *canonical commutation relations*.) Show that the same relations are satisfied by P and Q:

$$\{P, P\} = \{Q, Q\} = 0 \qquad ; \qquad \{Q, P\} = 1\,.$$

(d) More generally, show that the Poisson bracket between any two functions $A(p, q)$, $B(p, q)$ is unaffected by the change of variables

$$\{A, B\}_{P,Q} = \{A, B\}_{p,q}$$

where the notation means that on the left hand side we view $A(p, q)$ and $B(p, q)$ as functions of P, Q via the dependence of p and q on P, Q, and we define the P, Q Poisson bracket to be

$$\{A, B\}_{P,Q} = \frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} - \frac{\partial A}{\partial P}\frac{\partial B}{\partial Q}$$

while the p, q Poisson bracket is

$$\{A, B\}_{p,q} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

now with A and B viewed as functions of p, q. The previous questions was the particular case in which $A = Q$, $B = P$.

11. A pendulum consists of a light straight spring of length r which makes an angle θ to the vertical, at the end of which is connected a bob of mass m. If the natural length of the spring is r_0, the Lagrangian for this system can be written

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos(\theta) - \frac{k}{2}(r - r_0)^2.$$

Explain where each of the terms in this Lagrangian comes from. Find the Hamiltonian for this system and write down the equations of motion in Hamiltonian form. If $g = 0$ show that p_θ is conserved.

12. A pendulum consists of a weightless rod and a heavy bob. Initially it is at rest in vertical stable equilibrium. The upper end is then made to move down a straight line of slope α (with the horizontal) with constant acceleration f. Show that in the subsequent motion, the pendulum just becomes horizontal if $g = f(\cos\alpha + \sin\alpha)$.

13. A particle moves in the xy-plane subject to the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\Omega}{2}(-\dot{x}y + \dot{y}x),$$

where Ω is a constant.

(a) Write down the Lagrangian equations of motion.

(b) Show that the z-component of the usual angular momentum expression, $J_z = x\dot{y} - y\dot{x}$, about the origin is not (in general) conserved.

(c) Show that the z-component of the generalised angular momentum, $\mathcal{J}_z = xP_y - yP_x$ is conserved everywhere.

(d) Show that, for any solution of the equations of motion, there is a fixed point (call it A) such that the z-component of angular momentum J_z about A is conserved.

(e) Find the Hamiltonian for the system and show that it is conserved.

14. A bead of mass m slides, without friction, on a circular hoop of radius a. The hoop lies in a vertical plane which is constrained to rotate about a vertical diameter with constant angular speed ω. Choosing θ to be the angle between the bottom end of the vertical diameter and the diameter through the bead, show that

$$\frac{1}{2}ma^2\left(\dot{\theta}^2 - \omega^2\sin^2\theta\right) - mga\cos\theta$$

is constant during the motion.

15. In this question the notation is slightly more elaborate than usual: the n generalised coordinates will be written with superscripts q^i and then the Lagrangian is

$$L(q^i, \dot{q}^i) = \frac{1}{2}g_{ij}(q^k)\dot{q}^i\dot{q}^j,$$

where the n^2 functions g_{ij} of all the coordinates form a matrix which is symmetric and nonsingular. The *inverse* of the matrix with elements g_{ij} is a symmetric matrix whose elements we write with *superscripts* g^{ij}. Define the generalised momenta p_i as usual (using a subscript) and write an expression for the Hamiltonian. Write out Hamilton's equations. Verify that they are equivalent to Lagrange's equations.

16. A particle of mass m slides under gravity on a smooth parabolic wire with the shape $z = a^2x^2/2$, where the x-axis is horizontal and the z-axis points up. The wire is made to rotate about the z-axis with constant angular velocity Ω. Work in cylindrical polars (ρ, θ, z). Show that the points on the spinning paraboloid satisfy $(\rho, \theta, z) = (\rho, \Omega t, a^2\rho^2/2)$. By expressing the kinetic energy in terms of cylindrical polars show that the Lagrangian is

$$\frac{m}{2}\left(\dot{\rho}^2\left(1 + a^4\rho^2\right)\right) + \frac{m\rho^2}{2}\left(\Omega^2 - ga^2\right) .$$

Show that the Hamiltonian is

$$H = \frac{p^2}{2m(1 + a^4\rho^2)} + \frac{m\rho^2}{2}(ga^2 - \Omega^2).$$

where ρ is the cylindrical polar co-ordinate.

17. Introduce the totally antisymmetric Levi-Civita symbol ε_{ijk} defined by

$$\varepsilon_{ijk} = -\varepsilon_{jik} = -\varepsilon_{ikj}$$

and

$$\varepsilon_{123} = 1$$

so that ε_{ijk} is $+1$ if $(ijk) \in \{(123), (231), (312)\}$, -1 if $(ijk) \in \{(132), (321), (213)\}$, and 0 otherwise. Define the angular momentum generators by

$$J_i = \sum_{jk} \varepsilon_{ijk} x_j p_k .$$

(a) Show explicitly that the transformation generated by J_i leaves x_i fixed, and acts as an infinitesimal rotation on the plane defined by the other two coordinates:

$$\delta x_m = \epsilon \sum_j \varepsilon_{ijm} x_j$$

with ϵ an infinitesimal parameter. In particular, choosing $i = 1$, show that this is the infinitesimal form of a rotation in the (x_2, x_3) plane.

(b) Assume that we have a Hamiltonian of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + V(x_1^2 + x_2^2 + x_3^2)$$

with $V(s)$ an arbitrary function. Show that the angular momenta J_i are conserved for such Hamiltonians.

18. Consider the angular momenta J_i defined as in the previous question.

    (a) Show that

    $$\{J_i, J_j\} = \sum_k \varepsilon_{ijk} J_k \,.$$

    Hint: You might want to use that

    $$\sum_i \varepsilon_{iab}\varepsilon_{imn} = \delta_{am}\delta_{bn} - \delta_{an}\delta_{bm} \,.$$

    (b) Construct the total angular momentum

    $$J^2 = \sum_{i=1}^{3} J_i^2 \,.$$

    Show, using only algebraic properties of the Poisson bracket and the result $\{J_i, J_j\} = \sum_k \varepsilon_{ijk} J_k$ in the previous question, that if the J_i are conserved quantities, J^2 is also conserved. Show also that

    $$\{J^2, J_i\} = 0 \,.$$

19. In $2n$-dimensional phase space with Hamiltonian

    $$H = \frac{1}{2}\Sigma_{i=1}^{n}(p_i^2 + q_i^2),$$

    show that $M_{jk} \equiv p_j p_k + q_j q_k$ and $L_{jk} \equiv p_j q_k - q_j p_k$ are constants of the motion by evaluating the Poisson brackets $\{M_{jk}, H\}$, $\{L_{jk}, H\}$.