Lagrangian Mechanics I - Generalised Coordinates

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1. Determine how many degrees of freedom the following systems of particles have (i.e. what is the dimension of the configuration space). Give examples of possible generalised coordinates in the configuration space.

   (a) 3 particles moving freely in three-dimensional space.

   (b) 3 charged particles moving in three-dimensional space.

   (c) 3 particles in three dimensions, where particle 1 and particle 3 are a fixed distance from particle 2 but not from each other.

   (d) 3 particles in three dimensions which are all a fixed distance away from each other (this is like a rigid body).

   (e) 4 particles in three dimensions which are all a fixed distance away from each other.

   (f) 5 particles in three dimensions which are all a fixed distance away from each other.

   (g) A particle constrained to move on the surface of a two-dimensional bowl (the bowl could be defined by $z = x^2 + y^2$, $z \in [0, 1]$ in $\mathbb{R}^3$, for instance).

   (h) A pencil whose ends are constrained to be touching the inside of the previous bowl (for this question you can take the pencil to be infinitesimally thin or not).

   (i) A flexible rope, with its two ends fixed to a wall a metre apart from each other.

2. You have two points $\vec{x}_0$ and $\vec{x}_1$ in $\mathbb{R}^3$, and $\vec{\gamma}(t)\colon [0, 1] \to \mathbb{R}^3$ is a path connecting the two points, with $\vec{\gamma}(0) = \vec{x}_0$ and $\vec{\gamma}(1) = \vec{x}_1$. We denote the components of $\vec{\gamma}(t)$ by $\gamma_i(t)$. First argue that the total length of the path is

$$\text{Length}[\vec{\gamma}] = \int_0^1 dt \sqrt{\sum_{i=1}^{3} \dot{\gamma}_i(t)^2} \,.$$

   Using this, and the variational principle, show that a straight line between $\vec{x}_0$ and $\vec{x}_1$ minimizes the total length of $\vec{\gamma}$. *[Hint:] It might help to choose a parametrization of the curve well suited for straight lines.*

3. We have seen in the lectures that the Euler-Lagrange equation for a system with a single degree of freedom $x(t)$ and Lagrangian $L(x(t), \dot{x}(t))$ is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \,.$$

Show that for a system with N degrees of freedom, described by generalised coordinates q_i, $i \in 1, \ldots, N$ and Lagrangian $L(q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots, \dot{q}_N(t))$ the variational equation

$$\delta S = 0$$

implies the N equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0\,. \tag{1}$$

There is one equation for each value of i.

4. Show that if the Lagrangian depends on second order time derivatives $\ddot{x}(t)$, that is, if $L(x(t), \dot{x}(t), \ddot{x}(t))$, then the Euler-Lagrange equations become

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = 0\,.$$

(To derive these equations, assume that both x and $\dot{x}$ are held fixed at the endpoints.) More generally, what is the form of the Euler-Lagrange equations for Lagrangians depending on time derivatives of x of order k and lower? (Again, holding the first $k-1$ time derivatives fixed at the endpoints.)

5. The dynamics of a system with n degrees of freedom is specified by a Lagrangian $L(q, \dot{q}, t)$, where $q = (q_1, q_2, ..., q_n)$. For any function $f(q, t)$, show that

$$L' = L + \frac{df}{dt} = L + \sum_{k=1}^{n} \dot{q}_k \frac{\partial f}{\partial q_k} + \frac{\partial f}{\partial t}$$

produces the same equations of motion. Do it first using the Euler-Lagrange equations and then by showing that the variational principle is not affected by replacing L with L' in the action. You can assume that partial derivatives commute.

6. A particle of mass m is constrained to move on the paraboloid $z = x^2 + y^2$. Generalised coordinates q_1, q_2 are used to describe the system where

$$\begin{array}{rcl} x & = & q_1 \cos(q_2) \\ y & = & q_1 \sin(q_2) \\ z & = & q_1^2 \end{array}\,.$$

Find the kinetic energy of the particle in terms of q_1, q_2 and their time derivatives $\dot{q}_1, \dot{q}_2$.

7. A particle of mass m is constrained to move on the unit sphere $z^2 = 1 - x^2 - y^2$. Generalised coordinates q_1, q_2 are used to describe the system where

$$\begin{array}{rcl} x & = & q_1 \cos(q_2) \\ y & = & q_1 \sin(q_2) \end{array}\,.$$

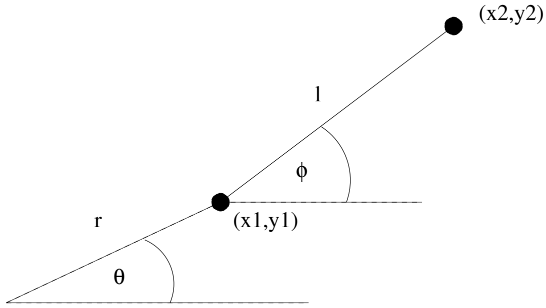

Figure 1: Configuration in problem 8

Find an expression for z in terms of q_1 and q_2, and hence find the kinetic energy of the particle in terms of q_1 and q_2. Describe the path of the particle for the particular solution $q_1 = \cos(\omega t)$, $q_2 = 0$, and use the expression you have found to calculate the value of the kinetic energy for this path.

8. Two particles of equal mass m are fastened to the ends of a light rigid rod of length l and can move freely in a two-dimensional plane. Determine the number of degrees of freedom of the system. Write an expression for the total kinetic energy of the system in terms of r, θ and ϕ,where r and θ are the polar coordinates of one of the masses and ϕ is the angle between the rod and the x-axis, as in figure 1.