Lagrangian Mechanics
Symmetries and conservation laws I

October 25, 2020

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1. For each of the following Lagrangians, state explicitly which coordinates are ignorable, and write down the corresponding conserved quantities:

   (a) $L = \frac{1}{w^2}\left(\dot{w}^2 + z^2\dot{\theta}^2\right) - \frac{1}{2}\dot{z}^2 + \sqrt{z^2 + w^2}$

   (b) $L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) - mr\cos(\theta)$

   (c) $L = \frac{1}{2}\left(\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\right)$.

2. A system is described by the Lagrangian

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}q_1^2\left(\dot{q}_2^2 + \dot{q}_3^2\right) - \frac{1}{2}(q_2^2 + q_3^2).$$

   (a) Show that $q_1 \to q_1$, $q_2 \to q_2 + \epsilon q_3$, $q_3 \to q_3 - \epsilon q_2$ is a symmetry of this Lagrangian and use Noether's Theorem to find a conserved quantity Q associated with this symmetry.

   (b) Verify, by taking the time derivative, that $\frac{dQ}{dt} = 0$ for the conserved charge that you found. You will need to use the Euler-Lagrange equations of motion.

3. A system is described by the Lagrangian

$$L = \left(\frac{\dot{x}}{x}\right)^2 + \left(\frac{\dot{y}}{y}\right)^2 + \ln(x)\frac{\dot{y}}{y} - \ln(y)\frac{\dot{x}}{x}.$$

   (a) Show that the transformation

$$x \to x + \epsilon x\ ,\ y \to y,$$

   is a symmetry, and use Noether's theorem to find the associated conserved quantity Q_x.

   (b) Find a similar transformation that leads to a second conserved quantity Q_y.

   (c) Solve the system for $x(t)$, $y(t)$ if initially at $t = 0$, $x = 2$, $y = 1$, $\dot{x} = 0$, and $\dot{y} = 1$. *[Hint: You might want to use the conserved quantities you found above.]*

4. The Lagrangian of a system is given by

$$L = -m\sqrt{\dot{w}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$$

(a) Show that the transformation

$$\begin{aligned} w &\rightarrow \cosh(\theta)w + \sinh(\theta)x \\ x &\rightarrow \cosh(\theta)x + \sinh(\theta)w \end{aligned}$$

where θ is a constant, leaves the Lagrangian invariant.

(b) Find the infinitesimal form of this transformation by taking θ to be an infinitesimal parameter ϵ, and only keeping terms to $O(\epsilon)$. Find the conserved charge Q associated with this transformation by using Noether's Theorem.

5. A system is described by a Lagrangian $L = L(q_1, .., q_n, \dot{q}_1, .., \dot{q}_n)$ which does not explicitly depend on time. Use the Euler-Lagrange equations to show directly that

$$E = \left(\sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) - L$$

is conserved (that is, $\frac{dE}{dt} = 0$). If in a particular case the Lagrangian is

$$L = \frac{1}{2}\dot{u}^2 + \frac{1}{2u^2}(\dot{v}^2 - 1)$$

find this conserved quantity. Find a second conserved quantity corresponding to the ignorable coordinate v.

6. Find the value for the conserved quantity E for each of the following Lagrangian systems:

(a) $L = \frac{1}{w^2}\left(\dot{w}^2 + z^2\dot{\theta}^2\right) - \frac{1}{2}\dot{z}^2 + \sqrt{z^2 + w^2}$

(b) $L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) - mr\cos(\theta)$

(c) $L = \frac{1}{2}\left(\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\right)$.