Lagrangian Mechanics
Symmetries and conservation laws II

October 31, 2022

---

1. Suppose that a particle of mass m falls under the influence of gravity. We turn on a fan below the particle, pushing the particle upwards, and increase the strength of the fan with time. We model this by a Lagrangian of the form

$$L = \frac{1}{2}m\dot{z}^2 - mgz + zf(t)$$

where z is the vertical coordinate and $f(t)$ is some function of time encoding the varying fan strength.

   (a) Write down the Euler-Lagrange equations for the system. (You do not need to solve them.)

   (b) Compute the energy of the system.

   (c) Verify, by taking the derivatives explicitly, that

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}$$

   for a solution of the Euler-Lagrange equations of motion.

2. In practice, we often assume the existence of a symmetry in order to constrain the form of the Lagrangian. Here is one simple example: assume that we have a Lagrangian of the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - ax^2 - by^2 - cy^3$$

for some unknown constants $a, b, c \in \mathbb{R}$. Working in (x, y) coordinates, find the relations that a, b and c need to satisfy so that arbitrary rotations around the origin of the x-y plane (that is, $x = y = 0$) are a symmetry of the Lagrangian with $F = 0$, or equivalently, so that arbitrary rotations by a small angle ϵ leave the Lagrangian invariant to first order in ϵ.

3. **[Solved during the second problems class, not for submission to gradescope.]** We now generalize the result of the previous problem. Let us switch to polar coordinates (r, θ), defined by

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

and assume that we have a Lagrangian of the form

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta)$$

for some function $V(r, \theta)$. Find the most general form of $V(r, \theta)$ compatible with arbitrary rotations around $r = 0$ being symmetries.