## Lagrangian Mechanics Symmetries and conservation laws II

November 2, 2024

1. Suppose that a particle of mass m falls under the influence of gravity. We turn on a fan below the particle, pushing the particle upwards, and increase the strength of the fan with time. We model this by a Lagrangian of the form

$$L = \frac{1}{2}m\dot{z}^2 - mgz + zf(t)$$

where z is the vertical coordinate and f(t) is some function of time encoding the varying fan strength.

- (a) Write down the Euler-Lagrange equations for the system. (You do not need to solve them.)
- (b) Compute the energy of the system.
- (c) Verify, by taking the derivatives explicitly, that

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}$$

for a solution of the Euler-Lagrange equations of motion you found in part (a).

2. In practice, we often assume the existence of a symmetry in order to constrain the form of the Lagrangian. Here is one simple example: assume that we have a Lagrangian of the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - ax^2 - by^2 - cy^3$$

for some unknown constants  $a, b, c \in \mathbb{R}$ . Working in (x, y) coordinates, find the relations that a, b and c need to satisfy so that arbitrary rotations around the origin of the x-y plane (that is, x = y = 0) are a symmetry of the Lagrangian with F = 0, or equivalently, so that arbitrary rotations by a small angle  $\epsilon$  leave the Lagrangian invariant to first order in  $\epsilon$ .

3. [Solved during the second problems class, not for submission to gradescope.] We now generalize the result of the previous problem. Let us switch to polar coordinates  $(r, \theta)$ , defined by

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

and assume that we have a Lagrangian of the form

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta)$$

for some function  $V(r, \theta)$ . Find the most general form of  $V(r, \theta)$  compatible with arbitrary rotations around r = 0 being symmetries.