# Lagrangian Mechanics
# Normal modes

November 7, 2020

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1. A system has a Lagrangian given by

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \ln(1 - x^2) + \cos(x + y).$$

   (a) Find the equations of motion.

   (b) Show that $x = 0, y = 0$ is a solution to the equations of motion.

   (c) By expanding the Lagrangian to quadratic order in $x$ and $y$ find an approximate Lagrangian $L_{\text{app}}$ for small $x$ and $y$. *Hint: You may use* $\log(1 + \epsilon) \approx \epsilon + \mathcal{O}(\epsilon^2)$.

   (d) Write the equations of motion arising from $L_{\text{app}}$ as

$$\ddot{\mathbf{q}} + \mathsf{A}\mathbf{q} = 0$$

   with $\mathbf{q} = (x, y)$, and $\mathsf{A}$ a constant $2 \times 2$ matrix.

   (e) Find the eigenvectors $\mathbf{v}^{(i)}$ of $\mathsf{A}$, with corresponding eigenvalues $\lambda^{(i)}$.

   (f) Check that

$$\mathbf{q} = \mathbf{v}^{(i)}(\alpha^{(i)} \cos(\omega^{(i)} t) + \beta^{(i)} \sin(\omega^{(i)} t))$$

   with $\alpha^{(i)}$ and $\beta^{(i)}$ arbitrary constants is a solution of the equations of motion as long as $\left(\omega^{(i)}\right)^2 = \lambda^{(i)}$.

   (g) The general solution of the system is then, by linearity,

$$\mathbf{q} = \sum_{i=1}^{2} \mathbf{v}^{(i)} \left[\alpha^{(i)} \cos\left(\omega^{(i)} t\right) + \beta^{(i)} \sin\left(\omega^{(i)} t\right)\right]$$

   If at $t = 0$ we have that $x = 0$, $y = 0$, $\dot{x} = 0.1$ and $\dot{y} = 0$, find the corresponding solution of the equations of motion for $L_{\text{app}}$ for all time.

2. A particle is moving in two dimensions, with standard kinetic term, under the action of a potential

$$V(x, y) = x^2 + (y^4 - 2y^2 + 1)$$

   (a) Identify all the stationary points of this potential, that is all the solutions of

$$\vec{\nabla} V := \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}\right) = (0, 0)\,.$$

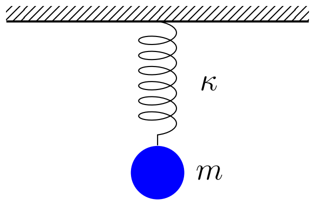
(b) Construct the matrix of second derivatives

$$\mathsf{A}_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$$

with $\mathbf{q} = (x, y)$, and evaluate it at each stationary point.

(c) Classify each stationary point as a minimum, maximum, or saddle point, depending on whether the eigenvalues of $\mathsf{A}$ at the point are all positive, all negative, or of mixed signs.

3. A spring of negligible mass, spring constant $\kappa$, and natural length $a$, hanging vertically with one end at $y = 0$, supports a particle of mass $m$, under the influence of a gravitational potential $V_{\text{gravity}} = mgy$.




(a) Show that the Lagrangian of the system is

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}\kappa(y + a)^2 - mgy$$

and write the Euler-Lagrange equations.

(b) Find the equilibrium position of the system (that is, a solution of the equations of motion that does not depend on time).

(c) Show that this equilibrium position is an extremum of the potential.

(d) Find the form of the generalized coordinate $q$ centered on this extremum such that the Lagrangian becomes

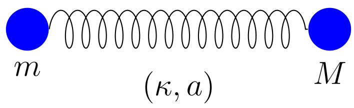
$$L = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}\kappa q^2 + c$$

for some constant $c$.

(e) Write the Euler-Lagrange equations for this system in the $q$ coordinate.

(f) Write the general solution for the motion of this system. Note that the kinetic term is not of canonical form, due to the presence of the mass $m$.

4. Two particles with masses $m$ and $M$, are connected by a spring (of spring constant $\kappa$ and natural length $a$) and are free to move on a straight line.




(a) Assuming that there are no frictional forces, show that the Lagrangian is

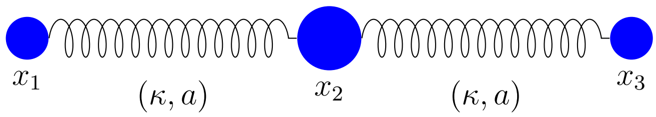
$$L = \frac{1}{2}(m\dot{q}_1^2 + M\dot{q}_2^2) - \frac{1}{2}\kappa(q_1 - q_2)^2,$$

where $q_1 = x_1$ and $q_2 = x_2 - a$, and $x_1, x_2$ are the positions of the particles along the line.

(b) Write the Euler-Lagrange equations for this system.

(c) Find the general solution to these equations of motion.

5. A very simple model of a triatomic molecule consists of three particles of unit mass linked with springs of spring constant $\kappa$ and natural length $a$, to form a linear molecule




Suppose the system is confined to moving on a straight line, with the positions of the particles being $x_1$, $x_2$ and $x_3$.

(a) Write the Lagrangian and Euler-Lagrange equations for the system, in the generalized coordinates $q_1 = x_1$, $q_2 = x_2 - a$ and $q_3 = x_3 - 2a$.

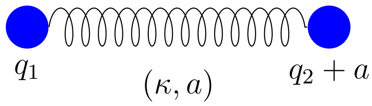
(b) Find the normal and zero modes for the system.

(c) Describe (by drawing a schematic picture, for instance) the behaviour of each normal mode. Assuming that I release the atoms from rest, starting from $(q_1, q_2, q_3) = (\delta, 0, -\delta)$, find the motion of the molecule in subsequent times.

6. A bowl is part of a surface $z = ax^2 + by^2 + 2hxy$ between $z = 0$ and $z = c > 0$, where $z$ is measured along the upward vertical. A particle slides smoothly under gravity inside the bowl. Find the periods of normal modes. If the particle is then constrained to slide so $y = kx$, find the period of the constrained motion and show that its greatest and least values (as $k$ varies) are the periods of the normal modes.

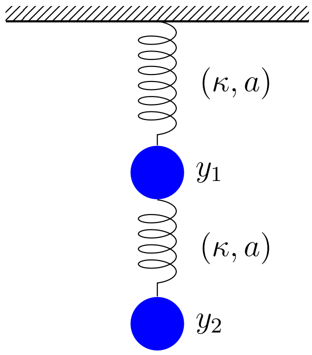
7. Two coupled pendula each hang from a fixed horizontal support, a distance $d$ apart. Each pendulum consists of a light rod of length $l$ with a heavy bob on the end, $M$ in one case, and a lighter $m$ in the other. The two bobs are connected by a light horizontal spring of spring constant $k$ and natural length $d$. Find the normal frequencies for this system and describe the normal modes. The pendula start from rest initially ($t = 0$) with the mass $M$ displaced a distance $a$, but $m$ in its equilibrium position. Show that the bounding value for the maximum amplitude of the mass $m$ in the subsequent oscillation is $2Ma/(M + m)$.

8. Two particles of unit mass are connected by a spring (of spring constant $\kappa$ and natural length $a$) and are free to move on a straight line. We introduce coordinates as follows:




   (a) Assuming that there are no frictional forces, show that the Lagrangian is
   $$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}\kappa(q_2 - q_1)^2,$$
   (b) Write down the general solution for the motion of the system.
   (c) Assuming that at $t = 0$ we have $(q_1, q_2) = (0, 0)$ and $(\dot{q}_1, \dot{q}_2) = (v, v)$ for some $v$, find the subsequent motion of the system.
   (d) Assume instead that at $t = 0$ we start from $(q_1, q_2) = (-c, c)$ for some $c$ and $(\dot{q}_1, \dot{q}_2) = (0, 0)$, find the subsequent motion of the system.

9. A spring of negligible mass, natural length $a$ and spring constant $\kappa$, hanging vertically with one end at $y = 0$, supports a particle of unit mass, and beneath it a second, identical spring carrying a second, identical unit mass.




   We assume that, in addition for the potential energy for the springs, there is a gravitational potential $V_{\text{gravity}} = gy_1 + gy_2$, pulling the particles towards negative values of $y$, where $y_1$ and $y_2$ are the vertical positions of the two particles. We introduce as generalised coordinates the vertical displacements $q_1$ and $q_2$ of the masses from their positions with the springs unextended (that is, at their natural length), which are at $y_1 = -a$ and $y_2 = -2a$. In other words, $q_1 = y_1 + a$ and $q_2 = y_2 + 2a$.

   (a) Write the Lagrangian for the system.
   (b) Find the position $(q_1^{(0)}, q_2^{(0)})$ of equilibrium (this is a solution of the equations of motion where $q_1$ and $q_2$ are constant in time).
   (c) By introducing new variables $r_1 = q_1 - q_1^{(0)}$ and $r_2 = q_2 - q_2^{(0)}$ measuring the displacement from equilibrium, find the normal modes and frequencies of vertical oscillations.