## Week 7 problems

1. Consider a system described by the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + 5\cos(q_1) + 2\sin(q_1q_2) + 2\cos(q_2).$$

- 1.1. Show that  $q_1(t) = q_2(t) = 0$  is a solution of the Euler-Lagrange equations of motion.
- 1.2. Construct an approximate Lagrangian describing small fluctuations around this solution.
- 1.3. Find the general solution for the Euler-Lagrange equations arising from this approximate Lagrangian.
- 1.4. Assume that at t = 0 the system is at rest (that is,  $\dot{q}_1(t = 0) = \dot{q}_2(t = 0) = 0$ ) and has position

$$\begin{pmatrix} q_1(t=0)\\ q_2(t=0) \end{pmatrix} = \begin{pmatrix} 1\\ 2 \end{pmatrix}$$

Find the solution of the approximate Lagrangian compatible with these initial conditions.

2. 2.1. Consider a field described by a Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t)^2 - \frac{1}{2}(u_x)^2 - \frac{1}{2}m^2u^2.$$

Find the equations of motion for u.

2.2. The resulting equation is still linear, and can be solved in terms of the normal modes:

$$u(x,t) = e^{i(\omega t \pm kx)}$$

Find the relation between  $\omega$ , k and m that needs to be satisfied so that this is a solution.

## [Problem 3 is on the next page]

- 3. Let f(x) be the function which is zero for  $|x| \ge b$  and equal to  $x^2 b^2$  for  $|x| \le b$ . An infinite string lies close to the x-axis and its displacement u(x,t) from it satisfies the wave equation,  $c^2 u_{xx} = u_{tt}$ .
  - 3.1. Assume the initial condition u(x,0) = f(x),  $u_t(x,0) = 0$ . Find the subsequent motion and illustrate your results with a sketch of u(x,t) for t = 0 and  $t \gg 0$ :
  - 3.2. Assume now the initial condition u(x,0) = 0  $u_t(x,0) = f(x)$ . Argue that the resulting motion looks like the following for small and large t:



You don't need to give the explicit solution, but you should explain the following features of the figure

- Why does it becomes constant in the central region for large times?
- What is the value of u(x,t) in the constant central region?
- For  $t \gg 0$ , how wide are the non-constant intervals on the edges?