Week 7 problems

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1. Consider a system described by the Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + 5\cos(q_1) + 2\sin(q_1 q_2) + 2\cos(q_2)\,.$$

   (a) Show that $q_1(t) = q_2(t) = 0$ is a solution of the Euler-Lagrange equations of motion.

   (b) Construct an approximate Lagrangian describing small fluctuations around this solution.

   (c) Find the general solution for the Euler-Lagrange equations arising from this approximate Lagrangian.

   (d) Assume that at $t = 0$ the system is at rest (that is, $\dot{q}_1(t=0) = \dot{q}_2(t=0) = 0$) and has position

$$\begin{pmatrix} q_1(t=0) \\ q_2(t=0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\,.$$

   Find the solution of the approximate Lagrangian compatible with these initial conditions.

2. (a) Consider a field described by a Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t)^2 - \frac{1}{2}(u_x)^2 - \frac{1}{2}m^2u^2\,.$$

   Find the equations of motion for u.

   (b) The resulting equation is still linear, and can be solved in terms of the normal modes:

$$u(x,t) = e^{i(\omega t \pm kx)}$$

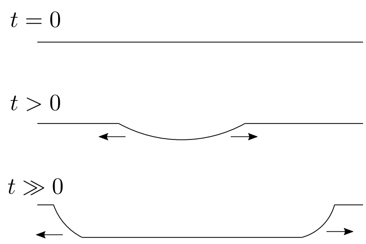
   Find the relation between ω, k and m that needs to be satisfied so that this is a solution.

[Problem 3 is on the next page]

3. Let $f(x)$ be the function which is zero for $|x| \geq b$ and equal to $x^2 - b^2$ for $|x| \leq b$. An infinite string lies close to the x-axis and its displacement $u(x,t)$ from it satisfies the wave equation, $c^2 u_{xx} = u_{tt}$.

   (a) Assume the initial condition $u(x,0) = f(x)$, $u_t(x,0) = 0$. Find the subsequent motion and illustrate your results with a sketch of $u(x,t)$ for $t = 0$ and $t \gg 0$:

   (b) Assume now the initial condition $u(x,0) = 0$ $u_t(x,0) = f(x)$. Argue that the resulting motion looks like the following for small and large t:



   You don't need to give the explicit solution, but you should explain the following features of the figure

   - Why does it becomes constant in the central region for large times?
   - What is the value of $u(x,t)$ in the constant central region?
   - For $t \gg 0$, how wide are the non-constant intervals on the edges?