Fields II

1. In the lectures we derived the one-dimensional wave equation $u_{tt}(x,t) = \frac{\tau}{\rho} u_{xx}(x,t)$, from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho(u_t)^2 - \frac{1}{2}\tau(u_x)^2$$

assuming that τ and ρ were constant. How is the one-dimensional classical wave equation modified when the vibrating string is inhomogeneous, i.e. when its density ρ and the tension τ are functions of x? Write the equation of motion in this case. Consider a potential solution of the form $u(x,t) = X(x)\cos(\omega t)$ and derive the differential equation satisfied by $X(x)$. Solve this equation for the special case in which τ and ρ are constant.

2. A string of finite length lies along the x-axis. Its subsequent displacement from the x-axis is given by $u(x,t)$. The string is fixed at both ends so that u obeys the boundary conditions $u(0,t) = u(\pi,t) = 0$. For simplicity we set $c^2 = 1$.

   Because of our choice of boundary conditions, we can alternatively think of an infinite string, subject to the usual wave equation $u_{tt} = u_{xx}$, in which we restrict the space of solutions to those of the form:

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(nx).$$

   where n is summed over the positive integers. From the equation of motion for u find the equation satisfied by each $b_n(t)$. Solve this equation, and in this way give the general solution for $u(x,t)$.

3. Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + m^2\cos(u)\,.$$

   The theory defined by this Lagrangian density is known as the **sine-Gordon** theory.

   (a) Compute the associated energy-momentum tensor.

   (b) An interesting peculiarity of this theory is that it admits travelling wave solutions with finite total energy that keep their shape as they travel. This is as in the example of the one-dimensional string we have considered so far in the lectures, but it is a quite non-generic phenomenon (such waves that travel without changing their shape are called **solitons**, and not every theory admits such solutions). A travelling wave solution would be of the form

$$u(x,t) = f(x - ct)$$

with c a constant, and $f(\zeta)$ some function of one variable. Find the equation satisfied by $f(\zeta)$.

(c) Show that

$$f(\zeta) = 4 \arctan\left(e^{\rho\zeta}\right)$$

is a solution of this equation of motion as long as

$$\rho^2 = \frac{m^2}{1-c^2}\,.$$

This implies that

$$u(x,t) = 4 \arctan\left(e^{\rho(x-ct)}\right)$$

is a solution to the equations of motion of the sine-Gordon theory.

Hint: You might use (without having to prove it) that

$$\sin(4\arctan(x)) = \frac{4(x-x^3)}{(1+x^2)^2}$$

and that

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

(d) Sketch how the solution we just found looks like.

4. Assume that we have a Lagrangian density $\mathcal{L}(u, u_x, u_t)$ depending on the field u and its time and space partial derivatives u_t and u_x. We define the energy-momentum tensor associated to $\mathcal{L}$ to be

$$T_{ij} \coloneqq u_i \frac{\partial \mathcal{L}}{\partial u_j} - \delta_{ij}\mathcal{L}\,.$$

Show, by directly taking the derivative, that T_{ij} is conserved, in the sense that

$$\sum_{j=0}^{1} \frac{\partial T_{ij}}{\partial x_j} = 0\,.$$

You will need to use the Euler-Lagrange equation for the field u

$$\frac{\partial \mathcal{L}}{\partial u} - \sum_{i=0}^{1} \frac{\partial}{\partial x_i}\left(\frac{\partial \mathcal{L}}{\partial u_i}\right) = 0\,,$$

where we have defined $x_0 \equiv t$ and $x_1 \equiv x$.

5. Consider a Lagrangian density given by the difference of kinetic and potential energies

$$\mathcal{L} = \mathcal{T}(u, u_t) - \mathcal{V}(u, u_x)$$

where

$$\mathcal{T} = (u_t)^2 f(u)$$

with $f(u)$ an arbitrary function of u, and $\mathcal{V}(u, u_x)$ an arbitrary function of u and u_x.

(a) Show that the purely time component of the energy-momentum tensor T_{tt} is equal to the energy density $\mathcal{E} = \mathcal{T} + \mathcal{V}$.

(b) Define the energy contained in an interval (a, b) by

$$E_{(a,b)}(t) := \int_a^b \mathcal{E}(x,t)\,dx\,.$$

Using the conservation equation you derived in the previous problem, find an expression for the energy flux $\mathcal{F}(x,t)$, in terms of $\frac{\partial \mathcal{V}}{\partial u_x}$ and u_t, such that

$$\frac{dE_{(a,b)}}{dt} = \mathcal{F}(a,t) - \mathcal{F}(b,t)\,.$$

This expression implies that the energy in the interval changes only due to the flux entering from the left and leaving from the right.

6. Imagine that we have a string, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho(u_t)^2 - \frac{1}{2}\tau(u_x)^2$$

extending from $x = -\infty$ to $x = 0$, where it ends. There are two classical sets of boundary conditions, known as "Dirichlet" and "Neumann", specifying what happens at $x = 0$. Let's treat them separately:

(a) For the Dirichlet boundary condition we set

$$u(0,t) = 0\,.$$

Show that the energy flux into the boundary at $x = 0$ vanishes. Consider a solution of the form

$$u(x,t) = \Re\left((e^{ikx} + Re^{-ikx})e^{-ikct}\right)$$

in which an incident right-moving "monochromatic" wave of unit amplitude gets reflected at the boundary. Determine which value of R is compatible with the Dirichlet boundary condition.

(b) Consider now D'Alembert's general solution to the wave equation without boundaries

$$u(x,t) = f(x-ct) + g(x+ct)\,.$$

for arbitrary functions f and g. Find the most general f and g that satisfy the Dirichlet boundary condition at $x = 0$. Sketch the form of the solution.

(c) For the Neumann one we instead set the boundary condition

$$u_x(0,t) = 0\,.$$

Show that the energy flux into the boundary at $x = 0$ also vanishes in this case. Assuming the same monochromatic wave ansatz

$$u(x,t) = \Re\left((e^{ikx} + Re^{-ikx})e^{-ikct}\right)$$

determine which value of R is compatible with the Neumann boundary condition.

(d) Find the most general solution to the wave equations with the Neumann boundary conditions, and sketch the solution.

7. In the case of the point particle we had that adding a total derivative to the Lagrangian

$$L \to L + \frac{dF(u,t)}{dt}$$

did not change the equations of motion. In the case of fields, the right statement is that the change in the Lagrangian density

$$\mathcal{L} \to \mathcal{L} + \frac{\partial F_1(u,x,t)}{\partial x} + \frac{\partial F_2(u,x,t)}{\partial t}$$

does not affect the Euler-Lagrange equations for u, for $F_1(u,x,t)$ and $F_2(u,x,t)$ arbitrary functions depending on u, x, t only (but not depending on u_x or u_t). Show that this is true by proving

$$\frac{\partial}{\partial u}\left(\frac{dF_1}{dx}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\left(\frac{dF_1}{dx}\right)}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\left(\frac{dF_1}{dx}\right)}{\partial u_t}\right) = 0$$

and similarly for $\frac{\partial F_2}{\partial t}$.

As a hint, you will need to derive the following generalisations of Lemma A in the notes (in appendix A)

$$\frac{\partial\left(\frac{\partial F(u,x,t)}{\partial t}\right)}{\partial u_t} = \frac{\partial\left(\frac{\partial F(u,x,t)}{\partial x}\right)}{\partial u_x} = \frac{\partial F(u,x,t)}{\partial u} \quad ; \quad \frac{\partial\left(\frac{\partial F(u,x,t)}{\partial t}\right)}{\partial u_x} = \frac{\partial\left(\frac{\partial F(u,x,t)}{\partial x}\right)}{\partial u_t} = 0$$

and Lemma B (same appendix)

$$\frac{\partial\left(\frac{\partial F(u,x,t)}{\partial x}\right)}{\partial u} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\right) \quad ; \quad \frac{\partial\left(\frac{\partial F(u,t)}{\partial t}\right)}{\partial u} = \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u}\right)\,.$$