Week 9 problems

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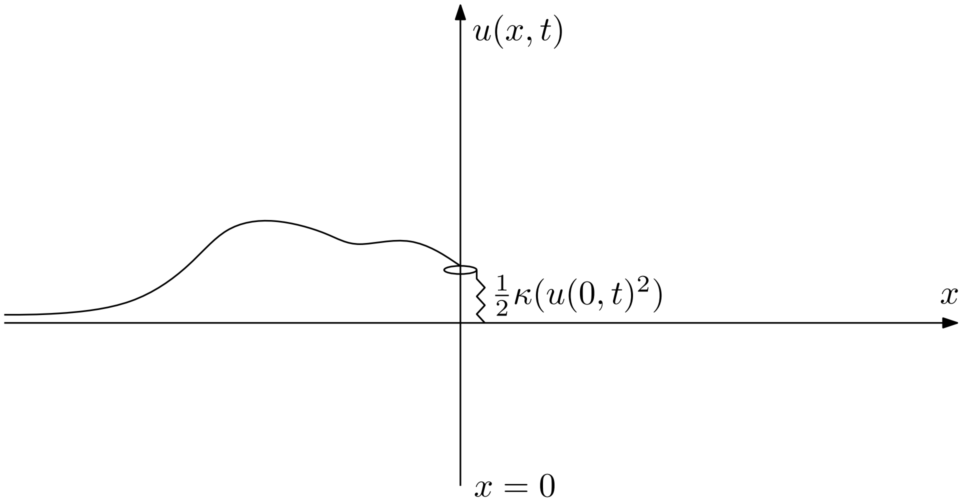
1. For a string obeying the wave equation $u_{tt} = u_{xx}$ (so we set $c^2 = 1$), we define a quantity
$$X(a,b) = \int_a^b \left((u_t)^3 + 3u_t(u_x)^2\right) dx$$
where $u_t = \frac{\partial u}{\partial t}$ and $u_x = \frac{\partial u}{\partial x}$. Use the wave equation to show that
$$\frac{d}{dt}X(a,b) = [g(u, u_t, u_x)]_a^b$$
for some function $g(u, u_t, u_x)$ which you should determine. Show that if u_t and u_x vanish at $x = \pm\infty$, then $X(-\infty, \infty)$ is a conserved quantitiy.

2. The motion of a string is governed by the wave equation $u_{tt} = u_{xx}$ (so we are setting $c^2 = 1$). The energy of the portion of the string between $x = a$ and $x = b$ is given by
$$E(a,b) = \frac{1}{2}\int_a^b \left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) dx.$$
   2.1. Use the wave equation to show that
$$\frac{d}{dt}E(a,b) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b.$$
   2.2. In the case that the solution u is of the form $u = f(x+t) - f(x-t)$, find the form of the energy in terms of f.
   2.3. Compute what $E(0,\infty)$ is in the case
$$u = \exp\left(-\frac{(x+t)^2}{2}\right) - \exp\left(-\frac{(x-t)^2}{2}\right)$$
   You should get that $E(0,\infty)$ a constant, whose value you should determine. (*It may be of help to recall that* $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$.)
   2.4. Use part (a) of this question to argue that $E(0,\infty)$ is a constant whenever f is even or odd, and is such that $\lim_{x\to\infty} f'(x) = 0$.

3. The Dirichlet and Neumann boundary conditions studied in the lectures are those in which the energy flowing into the boundary vanishes. A simple system in which energy can flow into the boundary can be constructed by assuming that we attach a spring to the one-dimensional string at the $x = 0$ boundary, as in the figure. The spring has constant κ and zero natural length, and it stores energy due to its extension, which is given by $u(0,t)$.



3.1. Write the equation encoding conservation of energy at the boundary.

3.2. Assuming the existence of a solution of the form

$$u(x,t) = \Re\left((e^{ipx} + Re^{-ipx})e^{-ipct}\right)$$

for the incident and reflected monochromatic waves, solve for R (as a function of p, κ and τ) by imposing energy conservation. [*Hint: as a consistency check, in the $\kappa \to \infty$ limit the incoming wave will not have enough energy to excite the spring, so in this limit you should have $R = -1$, the result for Dirichlet boundary conditions (recall problem 6 in the Week 8 problem sheet). Similarly in the $\kappa \to 0$ limit you should reproduce the result of Neumann boundary conditions, $R = 1$.*]