Hamiltonian Mechanics I - Hamilton's Equations and Poisson's Brackets

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1. (a) From the definition

$$\begin{aligned}\{B, A\} &= \frac{\partial B}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} \\ &= -\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right) = -\{A, B\}.\end{aligned}$$

(b) Again we have

$$\begin{aligned}\{aA + bC, B\} &= \frac{\partial(aA + bC)}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial(aA + bC)}{\partial p_i}\frac{\partial B}{\partial q_i} \\ &= a\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} + b\frac{\partial C}{\partial q_i}\frac{\partial B}{\partial p_i} - a\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - b\frac{\partial C}{\partial p_i}\frac{\partial A}{\partial q_i} \\ &= a\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right) + b\left(\frac{\partial C}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial C}{\partial p_i}\frac{\partial B}{\partial q_i}\right) \\ &= a\{A, B\} + b\{C, B\}.\end{aligned}$$

(c) From the product rule for differentiation we have

$$\begin{aligned}\{A, BC\} &= \frac{\partial A}{\partial q_i}\frac{\partial (BC)}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial (BC)}{\partial q_i} \\ &= \frac{\partial A}{\partial q_i}\left(\frac{\partial B}{\partial p_i}C + B\frac{\partial C}{\partial p_i}\right) - \frac{\partial A}{\partial p_i}\left(\frac{\partial B}{\partial q_i}C + B\frac{\partial C}{\partial q_i}\right) \\ &= \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)C + B\left(\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial C}{\partial q_i}\right) \\ &= \{A, B\}C + B\{A, C\}.\end{aligned}$$

2. The relations follow immediately from the definition of the Poisson bracket:

$$\begin{aligned}\{q_i, q_j\} &= \sum_k \left(\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}\right) \\ &= \sum_k (\delta_{ik} \cdot 0 - 0 \cdot \delta_{jk}) \\ &= 0\end{aligned}$$

and similarly for $\{p_i, p_j\}$:

$$\begin{aligned}\{p_i, p_j\} &= \sum_k \left(\frac{\partial p_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial p_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \\ &= \sum_k (0 \cdot \delta_{jk} - \delta_{ik} \cdot 0) \\ &= 0\end{aligned}$$

For the Poisson bracket between coordinates and momenta we have instead:

$$\begin{aligned}\{q_i, p_j\} &= \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \right) \\ &= \sum_k (\delta_{ik}\delta_{jk} - 0 \cdot 0) \\ &= \delta_{ij} \, .\end{aligned}$$

3. By the chain rule

$$\frac{dH}{dt} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial t}$$

and the result then follows from using Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} \qquad ; \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

4. That the Poisson bracket is conserved follows immediately from the Jacobi identity

$$\{\{A_1, A_2\}, H\} = -\{\{H, A_1\}, A_2\} - \{\{A_2, H\}, A_1\} = 0$$

since A_1 and A_2 are conserved.

For the rest, we start by proving the Lemma:

$$\begin{aligned}\frac{\partial}{\partial t}\{A, B\} &= \frac{\partial}{\partial t} \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right) \\ &= \sum_i \left(\frac{\partial}{\partial t} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right) - \frac{\partial}{\partial t} \left(\frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right) \right) \\ &= \sum_i \left(\left[\frac{\partial}{\partial t} \left(\frac{\partial A}{\partial q_i} \right) \right] \frac{\partial B}{\partial p_i} + \frac{\partial A}{\partial q_i} \left[\frac{\partial}{\partial t} \left(\frac{\partial B}{\partial p_i} \right) \right] - \left[\frac{\partial}{\partial t} \left(\frac{\partial A}{\partial p_i} \right) \right] \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial p_i} \left[\frac{\partial}{\partial t} \left(\frac{\partial B}{\partial q_i} \right) \right] \right) \\ &= \sum_i \left(\left[\frac{\partial}{\partial q_i} \left(\frac{\partial A}{\partial t} \right) \right] \frac{\partial B}{\partial p_i} + \frac{\partial A}{\partial q_i} \left[\frac{\partial}{\partial p_i} \left(\frac{\partial B}{\partial t} \right) \right] - \left[\frac{\partial}{\partial p_i} \left(\frac{\partial A}{\partial t} \right) \right] \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial p_i} \left[\frac{\partial}{\partial q_i} \left(\frac{\partial B}{\partial t} \right) \right] \right) \\ &= \left\{ \frac{\partial A}{\partial t}, B \right\} + \left\{ A, \frac{\partial B}{\partial t} \right\}\end{aligned}$$

With this in hand it is easy to prove what we want. Since A_1 and A_2 are conserved we have

$$\{A_1, H\} = -\frac{\partial A_1}{\partial t} \qquad ; \qquad \{A_2, H\} = -\frac{\partial A_2}{\partial t}\,.$$

So using Jacobi's identity

$$\begin{aligned}
\{\{A_1, A_2\}, H\} &= -\{\{H, A_1\}, A_2\} - \{\{A_2, H\}, A_1\} \\
&= \{\{A_1, H\}, A_2\} - \{\{A_2, H\}, A_1\} \\
&= -\left\{\frac{\partial A_1}{\partial t}, A_2\right\} + \left\{\frac{\partial A_2}{\partial t}, A_1\right\} \\
&= -\left(\left\{\frac{\partial A_1}{\partial t}, A_2\right\} + \left\{A_1, \frac{\partial A_2}{\partial t}\right\}\right) \\
&= -\frac{\partial}{\partial t}\{A_1, A_2\}
\end{aligned}$$

using the Lemma proven above. So indeed

$$\frac{d\{A_1, A_2\}}{dt} = \{\{A_1, A_2\}, H\} + \frac{\partial\{A_1, A_2\}}{\partial t} = 0\,.$$

5. The Lagrangian for a relativistic particle is

$$L = -mc\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2},$$

so the generalised momenta are given by

$$\begin{aligned}
p_x &= \frac{mc\dot{x}}{\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}} \\
p_y &= \frac{mc\dot{y}}{\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}} \\
p_z &= \frac{mc\dot{z}}{\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}}.
\end{aligned}$$

We need to invert these relations to find $\dot{x}$, $\dot{y}$, and $\dot{z}$ in terms of p_x, p_y and p_z. Squaring the three above equations and adding them together gives

$$p_x^2 + p_y^2 + p_z^2 = \frac{m^2c^2\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)}{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$$

which can be rearranged to give

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{c^2\left(p_x^2 + p_y^2 + p_z^2\right)}{m^2c^2 + p_x^2 + p_y^2 + p_z^2}$$

which now enables us to use the expression for the generalised momentum p_x to write

$$\begin{aligned}
\dot{x} &= \frac{p_x}{mc}\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} \\
&= \frac{p_x}{mc}\sqrt{c^2 - \frac{c^2\left(p_x^2 + p_y^2 + p_z^2\right)}{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} \\
&= \frac{p_x}{mc}\sqrt{\frac{m^2c^4}{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} \\
&= \frac{p_x c}{\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}}
\end{aligned}$$

and similarly we can write $\dot{y}$ and $\dot{z}$ by replacing p_x by p_y and p_z respectively in the numerator of the above expression. The Hamiltonian is defined to be

$$\begin{aligned}
H &= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L \\
&= \frac{c\left(p_x^2 + p_y^2 + p_z^2\right)}{\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} + mc\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} \\
&= \frac{c\left(p_x^2 + p_y^2 + p_z^2\right)}{\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} + mc\sqrt{c^2 - \frac{c^2\left(p_x^2 + p_y^2 + p_z^2\right)}{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} \\
&= \frac{c\left(p_x^2 + p_y^2 + p_z^2\right)}{\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} + mc\sqrt{\frac{m^2c^4}{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} \\
&= \frac{c\left(p_x^2 + p_y^2 + p_z^2\right) + m^2c^3}{\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}} \\
&= c\sqrt{m^2c^2 + p_x^2 + p_y^2 + p_z^2}.
\end{aligned}$$

If the 4-vector $p^\mu = (H/c, p_x, p_y, p_z)$ then

$$p^\mu p_\mu = \frac{H^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = \left(m^2c^2 + p_x^2 + p_y^2 + p_z^2\right) - p_x^2 - p_y^2 - p_z^2 = m^2c^2.$$

6. When the pulley is turned counterclockwise through an angle ϕ, the mass m_1 moves up ϕr whilst the mass m_2 moves down the same distance. So the kinetic energy is $(m_1 + m_2)r^2\dot{\phi}^2/2$, and the potential energy is $m_1 gr\phi - m_2 gr\phi$. Thus the Lagrangian is given by

$$L = \frac{(m_1 + m_2)}{2}r^2\dot{\phi}^2 - (m_1 - m_2)gr\phi.$$

The momentum is

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = (m_1 + m_2)r^2\dot{\phi}$$

and so

$$\dot{\phi} = \frac{p_\phi}{(m_1+m_2)r^2}$$

so the Hamiltonian is

$$\begin{aligned} H &= p_\phi \dot{\phi} - L \\ &= \frac{p_\phi^2}{(m_1+m_2)r^2} - \frac{p_\phi^2}{2(m_1+m_2)r^2} + (m_1-m_2)gr\phi \\ &= \frac{p_\phi^2}{2(m_1+m_2)r^2} + (m_1-m_2)gr\phi. \end{aligned}$$

Hamilton's equations of motions are

$$\begin{aligned} \dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{(m_1+m_2)r^2} \\ \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = (m_2-m_1)gr. \end{aligned}$$

The acceleration is

$$\begin{aligned} r\ddot{\phi} &= \frac{r\dot{p}_\phi}{(m_1+m_2)r^2} \\ &= \frac{(m_2-m_1)g}{(m_1+m_2)}. \end{aligned}$$

7. Somewhat monstrously

$$\begin{aligned} \{\{A,B\},C\} &= \left\{\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right), C\right\} \\ &= \frac{\partial}{\partial q_k}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)\frac{\partial C}{\partial p_k} - \frac{\partial}{\partial p_k}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)\frac{\partial C}{\partial q_k} \\ &= \frac{\partial^2 A}{\partial q_k \partial q_i}\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial p_k} + \frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial q_k \partial p_i}\frac{\partial C}{\partial p_k} - \frac{\partial^2 A}{\partial q_k \partial p_i}\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_k} + \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial q_k \partial q_i}\frac{\partial C}{\partial p_k} \\ &\quad - \frac{\partial^2 A}{\partial p_k \partial q_i}\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_k} - \frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial p_k \partial p_i}\frac{\partial C}{\partial q_k} + \frac{\partial^2 A}{\partial p_k \partial p_i}\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial q_k} + \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial p_k \partial q_i}\frac{\partial C}{\partial q_k}. \end{aligned}$$

Now all we need to do is to add the cyclic permutations of A, B, C together to get

$$\begin{aligned}
RHS &= \frac{\partial^2 A}{\partial q_k \partial q_i}\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial p_k} + \frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial q_k \partial p_i}\frac{\partial C}{\partial p_k} - \frac{\partial^2 A}{\partial q_k \partial p_i}\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_k} - \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial q_k \partial q_i}\frac{\partial C}{\partial p_k} \\
&\quad - \frac{\partial^2 A}{\partial p_k \partial q_i}\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_k} - \frac{\partial A}{\partial q_i}\frac{\partial^2 B}{\partial p_k \partial p_i}\frac{\partial C}{\partial q_k} + \frac{\partial^2 A}{\partial p_k \partial p_i}\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial q_k} + \frac{\partial A}{\partial p_i}\frac{\partial^2 B}{\partial p_k \partial q_i}\frac{\partial C}{\partial q_k} \\
&\quad \frac{\partial^2 B}{\partial q_k \partial q_i}\frac{\partial C}{\partial p_i}\frac{\partial A}{\partial p_k} + \frac{\partial B}{\partial q_i}\frac{\partial^2 C}{\partial q_k \partial p_i}\frac{\partial A}{\partial p_k} - \frac{\partial^2 B}{\partial q_k \partial p_i}\frac{\partial C}{\partial q_i}\frac{\partial A}{\partial p_k} - \frac{\partial B}{\partial p_i}\frac{\partial^2 C}{\partial q_k \partial q_i}\frac{\partial A}{\partial p_k} \\
&\quad - \frac{\partial^2 B}{\partial p_k \partial q_i}\frac{\partial C}{\partial p_i}\frac{\partial A}{\partial q_k} - \frac{\partial B}{\partial q_i}\frac{\partial^2 C}{\partial p_k \partial p_i}\frac{\partial A}{\partial q_k} + \frac{\partial^2 B}{\partial p_k \partial p_i}\frac{\partial C}{\partial q_i}\frac{\partial A}{\partial q_k} + \frac{\partial B}{\partial p_i}\frac{\partial^2 C}{\partial p_k \partial q_i}\frac{\partial A}{\partial q_k} \\
&\quad \frac{\partial^2 C}{\partial q_k \partial q_i}\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial p_k} + \frac{\partial C}{\partial q_i}\frac{\partial^2 A}{\partial q_k \partial p_i}\frac{\partial B}{\partial p_k} - \frac{\partial^2 C}{\partial q_k \partial p_i}\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_k} - \frac{\partial C}{\partial p_i}\frac{\partial^2 A}{\partial q_k \partial q_i}\frac{\partial B}{\partial p_k} \\
&\quad - \frac{\partial^2 C}{\partial p_k \partial q_i}\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_k} - \frac{\partial C}{\partial q_i}\frac{\partial^2 A}{\partial p_k \partial p_i}\frac{\partial B}{\partial q_k} + \frac{\partial^2 C}{\partial p_k \partial p_i}\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial q_k} + \frac{\partial C}{\partial p_i}\frac{\partial^2 A}{\partial p_k \partial q_i}\frac{\partial B}{\partial q_k} \\
&= 0
\end{aligned}$$

if you squint really hard, and remember that you can always swap the i and k indices since they are dummy indices.

8. The generalised momenta are given by $p_r = \dot{r}$ and $p_\theta = r^2\dot{\theta}$ so the Hamiltonian is given by

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = \dot{r}^2 + r^2\dot{\theta}^2 - L = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \mu\frac{\cos(\theta)}{r^2}.$$

Hamilton's equations of motion are

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = p_r \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2} \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{r^3} + 2\mu\frac{\cos(\theta)}{r^3} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \mu\frac{\sin(\theta)}{r^2}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{d}{dt}\left(p_\theta^2 + 2\mu\cos(\theta)\right) &= 2p_\theta\dot{p}_\theta - 2\mu\sin(\theta)\dot{\theta} \\
&= 2p_\theta\left(\mu\frac{\sin(\theta)}{r^2}\right) - 2\mu\sin(\theta)\left(\frac{p_\theta}{r^2}\right) = 0 \\
\Rightarrow p_\theta^2 + 2\mu\cos(\theta) &= \alpha
\end{aligned}$$

where α is constant. As H does not depend on time explicitly we expect it to be a constant E so

$$E = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \mu\frac{\cos(\theta)}{r^2} = \frac{\dot{r}^2}{2} + \frac{1}{2r^2}(\alpha - 2\mu\cos(\theta)) + \mu\frac{\cos(\theta)}{r^2} = \frac{\dot{r}^2}{2} + \frac{\alpha}{2r^2}.$$

9. The Hamiltonian H from L is given by $H = p_i\dot{q}_i - L$ with equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\ ,\ \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

where $p_i = \frac{\partial L}{\partial \dot{q}_i}$. The momenta from L' can be defined to be

$$P_i = \frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i} = p_i + \frac{\partial f}{\partial q_i}.$$

Correspondingly there is a new Hamiltonian

$$H' = P_i\dot{q}_i - L' = \left(p_i + \frac{\partial f}{\partial q_i}\right)\dot{q}_i - L - \frac{\partial f}{\partial q_i}\dot{q}_i - \frac{\partial f}{\partial t} = (p_i\dot{q} - L) - \frac{\partial f}{\partial t} = H - \frac{\partial f}{\partial t}.$$

The equations of motion for H' are

$$\dot{q}_i = \frac{\partial H'}{\partial P_i}\ ,\ \dot{P}_i = -\frac{\partial H'}{\partial q_i}.$$

Note that

$$H' = H(p, q) - \frac{\partial f(q,t)}{\partial t} = H(p(P,q), q) - \frac{\partial f(q,t)}{\partial t}$$

where we have used that $p_i = p_i(P, q) = P_i - \frac{\partial f(q,t)}{\partial q_i}$. So

$$\dot{q}_i = \frac{\partial H'}{\partial P_i} = \frac{\partial}{\partial P_i}\left(H(p(P,q), q) - \frac{\partial f(q,t)}{\partial t}\right) = \frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial P_i} = \frac{\partial H}{\partial p_k}\delta_{ki} = \frac{\partial H}{\partial p_i}$$

which is the same expression we obtained for $\dot{q}_i$ from the Hamilton equations of motion for H. Now turning our attention to the other equation

$$\begin{aligned}
\dot{P}_i &= -\frac{\partial H'}{\partial q_i} \\
&= -\frac{\partial}{\partial q_i}\left(H(p(P,q),q) - \frac{\partial f(q,t)}{\partial t}\right) \\
&= -\frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial q_i} - \frac{\partial H}{\partial q_i} + \frac{\partial^2 f}{\partial q_i \partial t} \\
&= \frac{\partial H}{\partial p_k}\frac{\partial^2 f}{\partial q_i \partial q_k} - \frac{\partial H}{\partial q_i} + \frac{\partial^2 f}{\partial q_i \partial t} \\
&= \dot{q}_k \frac{\partial^2 f}{\partial q_i \partial q_k} - \frac{\partial H}{\partial q_i} + \frac{\partial^2 f}{\partial q_i \partial t} \\
&= \frac{d}{dt}\left(\frac{\partial f}{\partial q_k}\right) - \frac{\partial H}{\partial q_i} \\
\Rightarrow \frac{d}{dt}\left(P_i - \frac{\partial f}{\partial q_k}\right) &= -\frac{\partial H}{\partial q_i} \\
\Rightarrow \dot{p}_i &= -\frac{\partial H}{\partial q_i}
\end{aligned}$$

which again is equivalent to the equation of motion derived from H. Notice that this is an example of a *canonical transformation.* The new momentum P_i is a mixture of the old coordinates and momenta, but together with a new Hamiltonian H' gives an equivalent physical description of the system.

10. (a) We start by inverting the given relations in order to express p, q in terms of P, Q. We have:

$$p = P\cos(\alpha) + Q\sin(\alpha)$$
$$q = -P\sin(\alpha) + Q\cos(\alpha)$$

In the particular case of $\alpha = \frac{\pi}{2}$ this is

$$p = Q$$
$$q = -P$$

so we are exchanging momentum with position (up to a sign). In these new coordinates the Hamiltonian becomes

$$H(P,Q) = \frac{1}{2m}Q^2 + \frac{1}{2}m\omega^2 P^2 .$$

The resulting equations of motion are

$$\dot{Q} = \frac{\partial H(P,Q)}{\partial P} = m\omega^2 P \tag{1}$$
$$\dot{P} = -\frac{\partial H(P,Q)}{\partial Q} = -\frac{1}{m}Q \tag{2}$$

Undoing the change in coordinates this implies

$$\dot{p} = -m\omega^2 q = -\frac{\partial H(p,q)}{\partial q}$$
$$\dot{q} = \frac{1}{m}p = \frac{\partial H(p,q)}{\partial p}$$

(b) By the chain rule:

$$\frac{\partial H}{\partial q} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q} = \frac{\partial H}{\partial Q}\cos(\alpha) - \frac{\partial H}{\partial P}\sin(\alpha)$$
$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p} = \frac{\partial H}{\partial Q}\sin(\alpha) + \frac{\partial H}{\partial P}\cos(\alpha)$$

On the other hand, from the expression of q and p in terms of Q and P we found above

$$-\dot{p} = -\dot{P}\cos(\alpha) - \dot{Q}\sin(\alpha)$$
$$\dot{q} = -\dot{P}\sin(\alpha) + \dot{Q}\cos(\alpha)$$

If we now impose Hamilton's equations for p, q we find

$$-\dot{P}\cos(\alpha) - \dot{Q}\sin(\alpha) = \frac{\partial H}{\partial Q}\cos(\alpha) - \frac{\partial H}{\partial P}\sin(\alpha)$$
$$-\dot{P}\sin(\alpha) + \dot{Q}\cos(\alpha) = \frac{\partial H}{\partial Q}\sin(\alpha) + \frac{\partial H}{\partial P}\cos(\alpha)$$

which for any α is equivalent to

$$-\dot{P} = \frac{\partial H}{\partial Q}$$
$$\dot{Q} = \frac{\partial H}{\partial P}$$

(c) Clearly $\{P,P\} = \{Q,Q\}$ by the antisymmetry of the Poisson bracket. For the remaining relation we use linearity:

$$\begin{aligned}\{Q,P\} &= \{q\cos(\alpha) + p\sin(\alpha), -q\sin(\alpha) + p\cos(\alpha)\} \\ &= \{q,p\}\cos^2(\alpha) - \{p,q\}\sin^2(\alpha) \\ &= \{q,p\}(\cos^2(\alpha) + \sin^2(\alpha)) \\ &= \{q,p\} \\ &= 1\end{aligned}$$

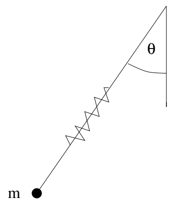

Figure 1: Set up for Question 8

(d) We have

$$\{A,B\}_{P,Q} = \frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} - \frac{\partial A}{\partial P}\frac{\partial B}{\partial Q}$$

Using the chain rule, the first term is

$$\begin{aligned}\frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} &= \left(\frac{\partial A}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial Q}\right)\left(\frac{\partial B}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial B}{\partial p}\frac{\partial p}{\partial P}\right)\\ &= \left(\frac{\partial A}{\partial q}\cos(\alpha) + \frac{\partial A}{\partial p}\sin(\alpha)\right)\left(\frac{\partial B}{\partial q}(-\sin(\alpha)) + \frac{\partial B}{\partial p}\cos(\alpha)\right)\end{aligned}$$

and similarly

$$\begin{aligned}\frac{\partial A}{\partial P}\frac{\partial B}{\partial Q} &= \left(\frac{\partial A}{\partial q}\frac{\partial q}{\partial P} + \frac{\partial A}{\partial p}\frac{\partial p}{\partial P}\right)\left(\frac{\partial B}{\partial q}\frac{\partial q}{\partial Q} + \frac{\partial B}{\partial p}\frac{\partial p}{\partial Q}\right)\\ &= \left(\frac{\partial A}{\partial q}(-\sin(\alpha)) + \frac{\partial A}{\partial p}\cos(\alpha)\right)\left(\frac{\partial B}{\partial q}\cos(\alpha) + \frac{\partial B}{\partial p}\sin(\alpha)\right)\end{aligned}$$

Combining both terms, and simplifying using $\cos^2(\alpha) + \sin^2(\alpha) = 1$ we obtain what we want, namely

$$\frac{\partial A}{\partial Q}\frac{\partial B}{\partial P} - \frac{\partial A}{\partial P}\frac{\partial B}{\partial Q} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

11. If we measure the angle θ to the downward vertical (see figure) then we see that the potential energy from gravity is $-mgr\cos(\theta)$. The spring potential energy is $k(r-r_0)^2/2$ and since we are simply working in polar coordinates the kinetic energy is given as usual. Putting this together we get

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos(\theta) - \frac{k}{2}(r-r_0)^2 .$$

The first step is to determine the conjugate momenta

$$\begin{aligned}p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r}\\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}.\end{aligned}$$

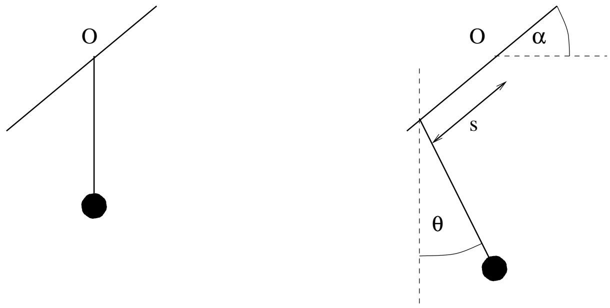

Figure 2: Set up for question 9

Working out the Hamiltonian we have

$$\begin{aligned} H &= p_\theta\dot{\theta} + p_r\dot{r} - L \\ &= \frac{p_\theta^2}{mr^2} + \frac{p_r^2}{m} - \left(\frac{m}{2}\left(\frac{p_r}{m}\right)^2 + \frac{mr^2}{2}\left(\frac{p_\theta}{mr^2}\right)^2 + mgr\cos(\theta) - \frac{k}{2}(r-r_0)^2\right) \\ &= \frac{p_\theta^2}{2mr^2} + \frac{p_r^2}{2m} - mgr\cos(\theta) + \frac{k}{2}(r-r_0)^2. \end{aligned}$$

Hamilton's Equations give

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + mg\cos(\theta) - k(r-r_0) \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = -mgr\sin(\theta). \end{aligned}$$

If $g = 0$ then p_θ is clearly conserved from the last equation.

12. The initial set up, and the position of the pendulum after it has started to slide down the slope is shown in the figure: Here I have chosen s, the distance down the slope that the pivot has gone, and θ the angle the pendulum makes with the vertical to describe the system. In terms of these coordinates the position of the mass is given by

$$\begin{aligned} x &= -s\cos(\alpha) + l\sin(\theta) \\ y &= -s\sin(\alpha) - l\cos(\theta) \end{aligned}$$

from which we deduce the velocity is given by

$$\begin{aligned} \dot{x} &= -\dot{s}\cos(\alpha) + l\cos(\theta)\dot{\theta} \\ \dot{y} &= -\dot{s}\sin(\alpha) + l\sin(\theta)\dot{\theta}. \end{aligned}$$

A little algebra gives that $\dot{x}^2 + \dot{y}^2 = \dot{s}^2 + l^2\dot{\theta}^2 - 2l\dot{s}\dot{\theta}\cos(\theta - \alpha)$. Combining this with the potential energy which is $mgy = -mg(s\sin(\alpha) + l\cos(\theta))$, we have that the Lagrangian for the system is

$$L = \frac{m}{2}\left(\dot{s}^2 + l^2\dot{\theta}^2 - 2l\dot{s}\dot{\theta}\cos(\theta - \alpha)\right) + mg(s\sin(\alpha) + l\cos(\theta)).$$

However, only θ is dynamical; s is prescribed by the condition that the pivot is made to move with constant acceleration f, that is $\ddot{s} = f$. Given that $s = \dot{s} = 0$ at time $t = 0$, we have that $s = ft^2/2$ and that $\dot{s} = ft$. Substituting these values into the Lagrangian we find that

$$L = \frac{m}{2}\left(f^2t^2 + l^2\dot{\theta}^2 - 2lft\dot{\theta}\cos(\theta - \alpha)\right) + mg(\frac{ft^2}{2}\sin(\alpha) + l\cos(\theta)).$$

Since this question finds its way onto a Hamiltonian sheet, we might as well solve it as a Hamiltonian system. Note that we only have one degree of freedom, θ. The corresponding momentum is given by

$$p = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - mlft\cos(\theta - \alpha).$$

The Hamiltonian is given by $p\dot{\theta} - L$, and even though it is NOT conserved (as it depends explicitly on t) we can still use the 'dots' rule to tell us that

$$\begin{aligned} H &= \frac{m}{2}l^2\dot{\theta}^2 - \frac{m}{2}f^2t^2 - mg(\frac{ft^2}{2}\sin(\alpha) + l\cos(\theta)) \\ &= \frac{m}{2}l^2\left(\frac{p + mlft\cos(\theta - \alpha)}{ml^2}\right)^2 - \frac{m}{2}f^2t^2 - mg(\frac{ft^2}{2}\sin(\alpha) + l\cos(\theta)). \end{aligned}$$

Hamilton's equations of motion are given by

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p} = \frac{p + mlft\cos(\theta - \alpha)}{ml^2} \\ \dot{p} &= -\frac{\partial H}{\partial \theta} = \left(\frac{p + mlft\cos(\theta - \alpha)}{ml^2}\right) mlft\sin(\theta - \alpha) - mgl\sin(\theta). \end{aligned}$$

Whilst we do not have conservation of energy, we do have that

$$\frac{dH}{dt} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

Thus

$$\begin{aligned} \frac{dH}{dt} &= \left(\frac{p + mlft\cos(\theta - \alpha)}{ml^2}\right) mlf\cos(\theta - \alpha) - mf^2t - mgft\sin(\alpha) \\ &= \dot{\theta}mlf\cos(\theta - \alpha) - mf^2t - mgft\sin(\alpha) \\ &= \frac{d}{dt}\left(mlf\sin(\theta - \alpha) - \frac{m}{2}f^2t^2 - \frac{m}{2}gft^2\sin(\alpha)\right), \end{aligned}$$

from which it follows that

$$\begin{aligned}
Q &= H - mlf\sin(\theta-\alpha) + \frac{m}{2}f^2t^2 + \frac{m}{2}gft^2\sin(\alpha) \\
&= \frac{m}{2}l^2\left(\frac{p+mlft\cos(\theta-\alpha)}{ml^2}\right)^2 - mlf\sin(\theta-\alpha) - mgl\cos(\theta) \\
&= \frac{m}{2}l^2\dot\theta^2 - mlf\sin(\theta-\alpha) - mgl\cos(\theta)
\end{aligned}$$

is a conserved quantity. At t=0, we have $\theta = 0$, $\dot\theta = 0$ so we see that $Q = mlf\sin(\alpha) - mlg$. If it just reaches horizontal at a later time, then at that time $\theta = \pi/2$ and $\dot\theta = 0$. Putting this into the conservation equation we have

$$Q = mlf\sin(\alpha) - mlg = 0 - mlf\cos(\alpha) - 0$$

which can be rearranged to give $g = f(\cos(\alpha) + \sin(\alpha))$.

13. The Euler Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} &= 0 \\
\frac{d}{dt}\left(\dot x - \frac{\Omega}{2}y\right) - \frac{\Omega}{2}\dot y &= 0 \\
\ddot x &= \Omega\dot y \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot y}\right) - \frac{\partial L}{\partial y} &= 0 \\
\frac{d}{dt}\left(\dot y + \frac{\Omega}{2}x\right) + \frac{\Omega}{2}\dot x &= 0 \\
\ddot y &= -\Omega\dot x.
\end{aligned}$$

From this it is straightforward to calculate

$$\begin{aligned}
\frac{dJ_z}{dt} &= \frac{d}{dt}(x\dot y - y\dot x) = x\ddot y - y\ddot x = \Omega(-x\dot x - y\dot y) \\
&= \Omega(-x(\Omega y + c_1) - y(-\Omega x + c_2)) \\
&= \Omega(-xc_1 - yc_2) \neq 0
\end{aligned}$$

where we have integrated the equations of motion to find $\dot x = \Omega y + c_1$ and $\dot y = -\Omega x + c_2$. (Strictly we should solve for x, y. We have $\ddot x = -\Omega^2 x + \Omega c_2$, so that $x = \alpha\cos(\Omega t) + \beta\sin(\Omega t) + c_2/\Omega$. From this we see that $y = -\alpha\sin(\Omega t) + \beta\cos(\Omega t) - c_1/\Omega$. This gives

$$\frac{dJ_z}{dt} = \Omega(-c_1\alpha - c_2\beta)\cos(\Omega t) + \Omega(c_2\alpha - c_1\beta)\sin(\Omega t)$$

which is certainly not zero. Now

$$\begin{aligned} P_x &= \frac{\partial L}{\partial \dot{x}} = \dot{x} - \frac{\Omega}{2} y \\ P_y &= \frac{\partial L}{\partial \dot{y}} = \dot{y} + \frac{\Omega}{2} x \end{aligned}$$

from which we see that

$$\begin{aligned} \frac{d\mathcal{J}_z}{dt} &= \frac{d}{dt}\left(xP_y - yP_x\right) = \frac{d}{dt}\left(x(\dot{y} + \frac{\Omega}{2}x) - y(\dot{x} - \frac{\Omega}{2}y)\right) \\ &= \frac{dJ_z}{dt} + \frac{d}{dt}\left(\frac{\Omega}{2}\left(x^2 + y^2\right)\right) = \Omega(-x\dot{x} - y\dot{y}) + \frac{d}{dt}\left(\frac{\Omega}{2}\left(x^2 + y^2\right)\right) = 0. \end{aligned}$$

Let us denote the angular momentum about a point (a, b) by $J(a, b)$. Then

$$\begin{aligned} \frac{dJ(a,b)}{dt} &= \frac{d}{dt}\left[(x-a)\,\dot{y} - (y-b)\dot{x}\right] = \frac{dJ_z}{dt} - a\ddot{y} + b\ddot{x} \\ &= \Omega(-c_1\alpha - c_2\beta + \beta\Omega a - \alpha\Omega b)\cos(\Omega t) + \Omega(c_2\alpha - c_1\beta - \alpha\Omega a - \beta\Omega b)\sin(\Omega t). \end{aligned}$$

We can make this vanish for any α, β if we set $a = c_2/\Omega$ and $b = -c_1/\Omega$. Finally the Hamiltonian is given by (using the expression in the notes)

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}\left(P_x + \frac{\Omega}{2}y\right)^2 + \frac{1}{2}\left(P_y - \frac{\Omega}{2}x\right)^2.$$

As this does not depend explicitly on time, we know it is conserved automatically, but if you are not convinced note that

$$\frac{dH}{dt} = \dot{x}\ddot{x} + \dot{y}\ddot{y} = \dot{x}\left(\Omega\dot{y}\right) + \dot{y}\left(-\Omega\dot{x}\right) = 0.$$

14. Again we shall solve this as a Hamiltonian question. Working in spherical polars the Lagrangian for the system is

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) + mgr\cos(\theta),$$

where in this case $r = a$ is a constant and $\dot{\phi} = \omega$, so that L reduces to

$$L = \frac{m}{2}\left(a^2\dot{\theta}^2 + a^2\sin^2(\theta)\omega^2\right) + mga\cos(\theta).$$

The momentum is given by $p_\theta = ma^2\dot{\theta}$, so that

$$H = \frac{p_\theta^2}{2ma^2} - \frac{m}{2}a^2\sin^2(\theta)\omega^2 - mga\cos(\theta).$$

Conservation of H gives the result. It is worth pointing out that whilst we have a conserved energy H, it is not the sum of the kinetic energy and the potential energy of the particle. The term $\frac{m}{2}a^2\sin^2(\theta)\omega^2$ would need to come with a plus sign for it to be the kinetic energy. Because we have put $\dot{\phi}=\omega$, this term which originally came from the kinetic energy has 'lost its dots', and therefore behaves like a potential energy, swapping its sign when we work out the energy. Physically we should not be too alarmed by this; the circular hoop needs to be driven (by a motor?) to keep it going at a constant angular velocity. This motor can put energy into the bead. The 'miracle' is that as we still have time translation invariance for the Lagrangian (no explicit t) we still have an energy-like quantity which is conserved.

15. By definition

$$\begin{aligned} p_l &= \frac{\partial L}{\partial \dot{q}^l} = \frac{1}{2}g_{ij}\frac{\partial}{\partial \dot{q}^l}\left(\dot{q}^i\dot{q}^j\right) \\ &= \frac{1}{2}g_{ij}\left(\delta^i_l\dot{q}^j + \delta^j_l\dot{q}^i\right) = \frac{1}{2}g_{lj}\dot{q}^j + \frac{1}{2}g_{il}\dot{q}^i = g_{lj}\dot{q}^j \end{aligned}$$

where we have used the symmetry of g_{ij} and relabelled dummy indices. Multiplying both sides of the equation by the inverse matrix g^{il} we have

$$\begin{aligned} g^{il}g_{lj}\dot{q}^j &= g^{il}p_l \\ \delta^i_j\dot{q}^j &= \dot{q}^i = g^{il}p_l. \end{aligned}$$

So we have by definition that the Hamiltonian is

$$\begin{aligned} H &= p_i\dot{q}^i - L \\ &= p_i g^{il}p_l - \frac{1}{2}g_{ij}g^{il}p_l g^{jm}p_m \\ &= g^{il}p_ip_l - \frac{1}{2}\delta^m_i g^{il}p_lp_m = \frac{1}{2}g^{il}p_ip_l. \end{aligned}$$

The Hamilton equation for $\dot{q}^i$ recovers the equation $\dot{q}^i = g^{il}p_l$, whilst the other equation is

$$\begin{aligned} \dot{p}_k &= -\frac{\partial H}{\partial q^k} \\ &= -\frac{1}{2}\frac{\partial g^{il}}{\partial q^k}p_ip_l. \end{aligned}$$

To compare this with the second order Lagrangian equation note that

$$\begin{aligned} \dot{p}_k &= \frac{d}{dt}\left(g_{kj}\dot{q}^j\right) = -\frac{1}{2}\frac{\partial g^{il}}{\partial q^k}p_ip_l \\ &= -\frac{1}{2}\frac{\partial g^{il}}{\partial q^k}g_{ij}\dot{q}^j g_{lm}\dot{q}^m. \end{aligned}$$

This expression can be simplified by understanding the relationship between the derivatives of g_{ij} and its inverse. By definition $g_{ij}g^{jk} = \delta_i^k$ and differentiating both sides with respect to q^x and remembering that δ_i^k is a constant we find

$$\begin{aligned}\frac{\partial g_{ij}}{\partial q^x}g^{jk} + g_{ij}\frac{\partial g^{jk}}{\partial q^x} &= 0\\ \Rightarrow g_{ij}\frac{\partial g^{jk}}{\partial q^x}g_{km} &= -\frac{\partial g_{ij}}{\partial q^x}g^{jk}g_{km} = -\frac{\partial g_{ij}}{\partial q^x}\delta^j_m = -\frac{\partial g_{im}}{\partial q^x}.\end{aligned}$$

Using this we see that our equation of motion becomes

$$\begin{aligned}\frac{d}{dt}\left(g_{kj}\dot{q}^j\right) &= \frac{1}{2}\frac{\partial g_{im}}{\partial q^k}\dot{q}^i\dot{q}^m\\ \Rightarrow g_{kj}\ddot{q}^j + \frac{\partial g_{kj}}{\partial q^l}\dot{q}^l\dot{q}^j &= \frac{1}{2}\frac{\partial g_{im}}{\partial q^k}\dot{q}^i\dot{q}^m\\ \Rightarrow g_{kj}\ddot{q}^j + \frac{1}{2}\frac{\partial g_{kj}}{\partial q^l}\dot{q}^l\dot{q}^j + \frac{1}{2}\frac{\partial g_{kl}}{\partial q^j}\dot{q}^l\dot{q}^j - \frac{1}{2}\frac{\partial g_{lj}}{\partial q^k}\dot{q}^l\dot{q}^j &= 0.\\ \Rightarrow \ddot{q}^i + g^{ik}\left(\frac{1}{2}\frac{\partial g_{kj}}{\partial q^l}\dot{q}^l\dot{q}^j + \frac{1}{2}\frac{\partial g_{kl}}{\partial q^j}\dot{q}^l\dot{q}^j - \frac{1}{2}\frac{\partial g_{lj}}{\partial q^k}\dot{q}^l\dot{q}^j\right) &= 0\\ \Rightarrow \ddot{q}^i + \Gamma^i_{lj}\dot{q}^l\dot{q}^j &= 0\end{aligned}$$

where

$$\Gamma^i_{lj} = g^{ik}\left(\frac{1}{2}\frac{\partial g_{kj}}{\partial q^l} + \frac{1}{2}\frac{\partial g_{kl}}{\partial q^j} - \frac{1}{2}\frac{\partial g_{lj}}{\partial q^k}\right).$$

16. Working in cylindrical polar coordinates (ρ, θ, z) the equation of the spinning paraboloid will be $(\rho, \Omega t, a^2\rho^2/2)$. The kinetic energy is given by

$$\frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\Omega^2 + \left(a^2\dot{\rho}\rho\right)^2\right)$$

whilst the potential energy is simply $mgz = mga^2\rho^2/2$, so that the Lagrangian is

$$\begin{aligned}L &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\Omega^2 + \left(a^2\dot{\rho}\rho\right)^2\right) - \frac{1}{2}mga^2\rho^2\\ &= \frac{m}{2}\left(\dot{\rho}^2\left(1 + a^4\rho^2\right)\right) + \frac{m\rho^2}{2}\left(\Omega^2 - ga^2\right)\end{aligned}$$

There is only one dynamical variable ρ whose conjugate momentum is given by

$$p = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}\left(1 + a^4\rho^2\right).$$

The Hamiltonian is given by $H = p\dot{\rho} - L$, which gives

$$\begin{aligned}H &= \frac{m}{2}\left(\dot{\rho}^2\left(1 + a^4\rho^2\right)\right) + \frac{m\rho^2}{2}\left(ga^2 - \Omega^2\right)\\ &= \frac{p^2}{2m\left(1 + a^4\rho^2\right)} + \frac{m\rho^2}{2}\left(ga^2 - \Omega^2\right).\end{aligned}$$

17. (a) The infinitesimal action of J_i on x_m is computed via the Poisson bracket, and it is given by

$$\begin{aligned}\delta x_m &= \epsilon\{x_m, J_i\} \\ &= \epsilon \sum_{jk} \varepsilon_{ijk}\{x_m, x_j p_k\} \\ &= \epsilon \sum_{jk} \varepsilon_{ijk}(\{x_m, x_j\}p_k + x_j\{x_m, p_k\}) \\ &= \epsilon \sum_{jk} \varepsilon_{ijk} x_j \{x_m, p_k\} \\ &= \epsilon \sum_{jk} \varepsilon_{ijk} x_j \delta_{mk} \\ &= \epsilon \sum_{j} \varepsilon_{ijm} x_j \,.\end{aligned}$$

In particular, when $m = i$ this vanishes, since ε_{ijk} is totally antisymmetric. When $m \neq i$ this is a rotation. Choose for example $i = 1$. Then we have

$$\delta x_1 = 0 \qquad ; \qquad \delta x_2 = -\epsilon x_3 \qquad ; \qquad \delta x_3 = \epsilon x_2 \,.$$

On the other hand, a finite rotation in the plane (x_2, x_3) is given by

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

keeping x_1 fixed, which reproduces the transformation law above if we set $\alpha = \epsilon$ and expand to first order in ϵ.

(b) We can prove this by direct computation. We have

$$\begin{aligned}\frac{dJ_i}{dt} &= \{J_i, H\} + \frac{\partial J_i}{\partial t} \\ &= \{J_i, H\} \\ &= \sum_{jk} \varepsilon_{ijk}\{x_j p_k, H\} \\ &= \sum_{jk} \varepsilon_{ijk}(x_j\{p_k, H\} + \{x_j, H\}p_k) \\ &= \sum_{jk} \varepsilon_{ijk}(-x_j\{H, p_k\} - \{H, x_j\}p_k) \\ &= \sum_{jk} \varepsilon_{ijk}\left(-x_j \frac{\partial H}{\partial x_k} + p_k \frac{\partial H}{\partial p_j}\right) \\ &= \sum_{jk} \varepsilon_{ijk}\left(-x_j (2x_k) V'\left(\sum_n x_n^2\right) + p_j p_k\right) \\ &= 0\end{aligned}$$

where in the last step we have used $\sum_{jk} \varepsilon_{ijk} u_j u_k = 0$ for any vector u (due to antisymmetry of ε), and the chain rule acting on V. We have also used that for any function $f(p, q, t)$

$$\{f(p,q,t), p_i\} = \frac{\partial f}{\partial q_i} \qquad ; \qquad \{f(p,q,t), q_i\} = -\frac{\partial f}{\partial p_i}$$

in the particular case $f = H$.

18. (a) Let us start with the right hand side. We have

$$\begin{aligned}
\sum_k \varepsilon_{ijk} J_k &= \sum_{klm} \varepsilon_{ijk} \varepsilon_{klm} x_l p_m \\
&= \sum_{klm} \varepsilon_{kij} \varepsilon_{klm} x_l p_m \\
&= \sum_{lm} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jm}) x_l p_m \\
&= x_i p_j - x_j p_i \,.
\end{aligned}$$

For the left hand side:

$$\begin{aligned}
\{J_i, J_j\} &= \sum_{ablm} \varepsilon_{iab} \varepsilon_{jlm} \{x_a p_b, x_l p_m\} \\
&= \sum_{ablm} \varepsilon_{iab} \varepsilon_{jlm} (\{x_a p_b, x_l\} p_m + x_l \{x_a p_b, p_m\}) \\
&= \sum_{ablm} \varepsilon_{iab} \varepsilon_{jlm} (x_a \{p_b, x_l\} p_m + x_l \{x_a, p_m\} p_b)
\end{aligned}$$

Using that $\{x_i, x_j\} = \{p_i, p_j\} = 0$, as we saw in the first problem. Using now that $\{x_i, p_j\} = \delta_{ij}$, we can continue

$$\begin{aligned}
\{J_i, J_j\} &= \sum_{ablm} \varepsilon_{iab} \varepsilon_{jlm} (-x_a \delta_{bl} p_m + x_l \delta_{am} p_b) \\
&= \left(-\sum_{abm} \varepsilon_{iab} \varepsilon_{jbm} x_a p_m \right) + \left(\sum_{abl} \varepsilon_{iab} \varepsilon_{jla} x_l p_b \right) \\
&= \left(-\sum_{abm} \varepsilon_{bia} \varepsilon_{bmj} x_a p_m \right) + \left(\sum_{abl} \varepsilon_{abi} \varepsilon_{ajl} x_l p_b \right) \\
&= \left(-\sum_{am} (\delta_{im}\delta_{aj} - \delta_{am}\delta_{ij}) x_a p_m \right) + \left(\sum_{bl} (\delta_{bj}\delta_{il} - \delta_{bl}\delta_{ij}) x_l p_b \right) \\
&= \left(-x_j p_i + \delta_{ij} \sum_m x_m p_m \right) + \left(x_i p_j - \delta_{ij} \sum_b x_b p_b \right)
\end{aligned}$$

now, clearly $\sum_m x_m p_m = \sum_b x_b p_b$, so we conclude

$$\{J_i, J_j\} = x_i p_j - x_j p_i$$

which is indeed equal to $\sum_k \varepsilon_{ijk} J_k$, as we showed above.

(b) Using the fact that the Poisson bracket is bilinear, and that it obeys Leibniz Rule, we have

$$\{J^2, H\} = \sum_i 2J_i\{J_i, H\} = 0$$

since each of the J_i is separately conserved, by assumption. To show that the J_i have vanishing bracket with J^2 we compute

$$\begin{aligned}\{J_i, J^2\} &= \sum_j 2\{J_i, J_j\}J_j \\ &= \sum_{jk} 2\varepsilon_{ijk} J_j J_k \\ &= 0\end{aligned}$$

where the last relation follows from the antisymmetry of ε_{ijk}.

19. In index notation we can write $H = (p_i p_i + q_i q_i)/2$ so we have that

$$\begin{aligned}\{M_{jk}, H\} &= \left\{p_j p_k + q_j q_k, \frac{1}{2}(p_i p_i + q_i q_i)\right\} \\ &= \frac{1}{2}\left[\frac{\partial}{\partial q_l}(p_j p_k + q_j q_k)\frac{\partial}{\partial p_l}(p_i p_i + q_i q_i) - \frac{\partial}{\partial p_l}(p_j p_k + q_j q_k)\frac{\partial}{\partial q_l}(p_i p_i + q_i q_i)\right] \\ &= \frac{1}{2}\left[(\delta_{lj} q_k + \delta_{lk} q_j)\, 2p_l - (\delta_{lj} p_k + \delta_{lk} p_j)\, 2q_l\right] \\ &= q_k p_j + q_j p_k - p_k q_j - p_j q_k = 0.\end{aligned}$$

Similarly we have that

$$\begin{aligned}\{L_{jk}, H\} &= \left\{p_j q_k - q_j p_k, \frac{1}{2}(p_i p_i + q_i q_i)\right\} \\ &= \frac{1}{2}\left[\frac{\partial}{\partial q_l}(p_j q_k - q_j p_k)\frac{\partial}{\partial p_l}(p_i p_i + q_i q_i) - \frac{\partial}{\partial p_l}(p_j q_k - q_j p_k)\frac{\partial}{\partial q_l}(p_i p_i + q_i q_i)\right] \\ &= \frac{1}{2}\left[(\delta_{lk} p_j - \delta_{lj} p_k)\, 2p_l - (\delta_{lj} q_k - \delta_{lk} q_j)\, 2q_l\right] \\ &= p_k p_j - p_j p_k - q_j q_k + q_k q_j = 0.\end{aligned}$$