Lagrangian Mechanics I - Generalised Coordinates

1.

(a) 9 degrees of freedom (three numbers needed to specify position of each one, for example in Cartesian coordinates).

(b) 9 degrees of freedom (three numbers needed to specify position of each one). The fact they are charged means they feel forces, but does not restrict them from moving to any position so still need all 9 numbers to specify configuration. Cartesian coordinates are again a good choice for generalised coordinates.

(c) 7 degrees of freedom. There are now two constraints between the 3 particles which removes 2 of the degrees of freedom. Or alternatively, argue that you need 3 numbers to specify the position of particle 2, and as particles 1 and 3 must lie on the surface of spheres centred on particle 2 (since they are some fixed distance from particle 2), you need 2 further numbers each for each of particle 1 and 3 to specify their positions. Generalised coordinates could be position of particle 2 in Cartesian coordinates, and four angles, two for each sphere.

(d) In this case there is a further constraint than in part (c) coming from the fixed distance between 1 and 3, so there are 6 degrees of freedom. We could choose the coordinate of the centre of mass, and three Euler angles for the orientation.

(e) This is still a rigid body, so there are 6 degrees of freedom in all. Actually in this case can still argue that there are 6 independent constraints between all four particles (they are independent, since if I were to remove any side from the tetrahedron joining the four particles, one could then deform it), and one needs 12 numbers to specify the positions of the four particles. $12 - 6 = 6$.

(f) The answer is still 6 (for example, the centre of mass and three Euler angles). Note that in this case we have 15 coordinates specifying the positions of the particles, and 10 constraints. But not all of the constraints are independent: knowing the distance of any given particle to three others completely fixes its position (perhaps up to some discrete choices, but the point is that the position cannot change continuously any more), without having to consider any other constraints.

(g) If its on a two-dimensional bowl, you need two numbers to specify the position by definition. So 2 degrees of freedom. One could choose z and an angle along the symmetry axis, for instance, or simply x and y.

(h) For a very thin pencil, you need 2 coordinates to specify the position of one end on the bowl. I am now free to move the other end around in a sort of circle, so I need one further coordinate to specify where on the circle it is. Alternatively argue that need two numbers to specify the position of each end of the pencil on the two dimensional surface, but there is a constraint that the two points must be a fixed distance apart. So we get $4-1=3$ degrees of freedom. If the pencil is thick then it can rotate about its axis of symmetry, giving an extra degree of freedom, i.e. 4.

(i) The 'configuration' of the rope is specified as a curve in three dimensions (if we take the rope to be very thin). If we parametrise the curve using the parameter s we can take the curve to be $(x(s), y(s), z(s))$. If we take the ends of the rope to be at $s=0$ and $s=1$ respectively, then the endpoints of the rope are at $(x(0), y(0), z(0))$ and $(x(1), y(1), z(1))$ respectively. If the ends are a metre apart then

$$(x(1)-x(0))^2 + (y(1)-y(0))^2 + (z(1)-z(0))^2 = 1.$$

No matter really! The important point here is I need three FUNCTIONS $x(s)$, $y(s)$ and $z(s)$ to define the position of the rope. To specify a function you need to specify an infinite number of numbers, so the rope has an infinite number of degrees of freedom! Of course, you might like to argue that the rope is made up of finitely many atoms, each of which could ideally be described by the three coordinates of its position, and this would reduce the number of degrees of freedom to a finite (albeit rather huge) number. We shall see later on, its easier to deal with infinite degrees of freedom and the idealised functions $x(s)$, $y(s)$ and $z(s)$!

2. The given total length can be understood as the integral of the infinitesimal line element

$$ds = \sqrt{d\gamma_1^2 + d\gamma_2^2 + d\gamma_3^2}$$

where γ_i are the components of γ. Since the γ_i depend on t only, we have

$$d\gamma_i(t) = \frac{d\gamma_i}{dt} dt$$

which leads to the given total length. The total length is of the form $S = \int dt\, L$, with $L = \sqrt{\ldots}$ so to find the extremum we can use the Euler-Lagrange equations. In this case they are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}_i}\right) - \frac{\partial L}{\partial \gamma_i} = 0\,.$$

The Lagrangian does not depend on γ_i itself, so the second term vanishes. For a suitable choice of t, a straight line satisfies that

$$\gamma_i(t) = \gamma_i(0) + t(\gamma_i(1) - \gamma_i(0)) \tag{1}$$

and it is immediate to see that this satisfies the Euler-Lagrange equations.

This shows that straight paths are local extrema. By basic trigonometry they are also minima. (Consider for example a small triangle, with one edge the straight line we are after. The length of the other two edges is necessarily larger than the straight line itself.)

3. This is bookwork. For a slightly displaced $q + \delta q$ (where we denote $q = (q_1, \ldots, q_N)$) we have (to first order in displacements)

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \\
&= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) \\
&= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i + \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right) \\
&= \left[\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i \,.
\end{aligned}$$

The first term vanishes because we are only considering variations with fixed end-points. Vanishing of the second term requires, by the fundamental lemma of the calculus of variations, the N Euler-Lagrange equations that we are after, since the δq_i variations are independent (by assumption, since we are assuming that the q_i are generalised coordinates).

4. The derivation is as above, going one degree higher in the expansion. We will do for arbitrary generalised coordinates q_i.

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i \\
&= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) + \sum_{i=1}^{N} \frac{\partial L}{\partial \ddot{q}_i}\frac{d^2}{dt^2}(\delta q_i) \\
&= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i - \frac{d\delta q_i}{dt}\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) + \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i} + \delta \dot{q}_i \frac{\partial L}{\partial \ddot{q}_i}\right) \\
&= \left[\delta q_i \frac{\partial L}{\partial \dot{q}_i} + \delta \dot{q}_i \frac{\partial L}{\partial \ddot{q}_i} - \delta q_i \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\right)\delta q_i
\end{aligned}$$

and first term vanishes since $\delta q_i = \delta \dot{q}_i = 0$ at the endpoints. The general form is clearly

$$\sum_{i=0}^{n} (-1)^i \left(\frac{d}{dt}\right)^i \frac{\partial L}{\partial \left(\left(\frac{d}{dt}\right)^i q\right)} = 0$$

by a trivial generalisation of the argument above.

5. The equations of motion arising from L' are

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = 0$$

which in this particular case become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial f}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} - \sum_{k=1}^{n} \dot{q}_k \frac{\partial^2 f}{\partial q_i \partial q_k} - \frac{\partial^2 f}{\partial q_i \partial t} = 0.$$

Using that

$$\frac{d}{dt}\left(\frac{\partial f(q_k, t)}{\partial q_i}\right) = \sum_{k=1}^{n} \dot{q}_k \frac{\partial^2 f}{\partial q_i \partial q_k} + \frac{\partial^2 f}{\partial t \partial q_i},$$

we can see that the L' equations of motion becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

so that the L' equations of motion are equivalent to the L equations.

In terms of the action principle, we have

$$S' = \int_{t_i}^{t_f} L'\, dt = \int_{t_i}^{t_f} \left(L + \frac{df}{dt}\right) dt = S + (f(q(t_f), t_f) - f(q(t_i), t_i))$$

so that, as we vary the set of paths keeping $q(t_i)$ and $q(t_f)$ fixed we have

$$\delta S = \delta S'$$

since $f(q(t_f), t_f)$ and $f(q(t_i), t_i))$ are held fixed.

6. The kinetic energy is $m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/2$. From the expressions for x, y and z we see that

$$\begin{aligned} \dot{x} &= \dot{q}_1 \cos(q_2) - q_1 \sin(q_2)\dot{q}_2 \\ \dot{y} &= \dot{q}_1 \sin(q_2) + q_1 \cos(q_2)\dot{q}_2 \\ \dot{z} &= 2q_1\dot{q}_1 \end{aligned}$$

and so we have that the kinetic energy T is given by

$$\begin{aligned} T &= \frac{m}{2}\left[(\dot{q}_1 \cos(q_2) - q_1 \sin(q_2)\dot{q}_2)^2 + (\dot{q}_1 \sin(q_2) + q_1 \cos(q_2)\dot{q}_2)^2 + (2q_1\dot{q}_1)^2\right] \\ &= \frac{m}{2}\left[\dot{q}_1^2 + q_1^2\dot{q}_2^2 + 4q_1^2\dot{q}_1^2\right]. \end{aligned}$$

7. In this case we have $z^2 = 1 - x^2 - y^2 = 1 - q_1^2$ so that $z = \sqrt{1 - q_1^2}$. The expressions for x and y are just 2d polars so we can write that the K.E is

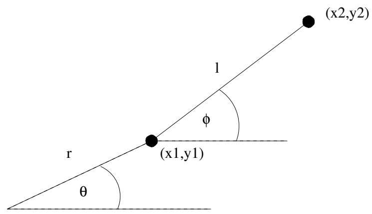
$$T = \frac{m}{2}\left[(\dot{x}^2 + \dot{y}^2) + \dot{z}^2\right] = \frac{m}{2}\left[(\dot{q}_1^2 + q_1^2\dot{q}_2^2) + \left(\frac{-q_1\dot{q}_1}{\sqrt{1 - q_1^2}}\right)^2\right] = \frac{m}{2}\left(\frac{\dot{q}_1^2}{1 - q_1^2} + q_1^2\dot{q}_2^2\right).$$

The path corresponding to $q_1 = \cos(\omega t)$, $q_2 = 0$ is given by $(x, y, z) = (\cos(\omega t), 0, \sin(\omega t))$. That is, the particle moves on a unit circle in the $x - z$ plane with constant angular velocity ω. The kinetic energy is given by

$$T = \frac{m}{2}\left(\frac{\dot{q}_1^2}{1-q_1^2} + q_1^2\dot{q}_2^2\right) = \frac{m}{2}\left(\frac{\sin^2(\omega t)\omega^2}{1-\cos(\omega t)^2}\right) = \frac{m}{2}\omega^2.$$

This is a constant, and nothing special happens at $t = 0$ as we might expect physically, as the particle is moving at constant velocity around the unit circle. Mathematically our expression appears to have a pole as $q_1 \to 1$ as $t \to 0$ but this cancels with the numerator! The apparent pole at $q_1 = 1$ is a coordinate artefact. Physically nothing singular happens there, its simply a byproduct of our choice of coordinates for the sphere.

8. Clearly there are three degrees of freedom (i.e. r, θ, and ϕ). The position of the two



masses are given by

$$\begin{aligned}(x_1,\, y_1) &= (r\cos(\theta)\,,\, r\sin(\theta))\\ (x_2\,,\, y_2) &= (r\cos(\theta) + l\cos(\phi)\,,\, r\sin(\theta) + l\sin(\phi)).\end{aligned}$$

The kinetic energy of the first mass is $m(\dot{r}^2 + r^2\dot{\theta}^2)/2$ as usual for 2d polars. The kinetic energy of the second particle is

$$\begin{aligned}\frac{m}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) &= \frac{m}{2}\left[\left(\dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta} - l\sin(\phi)\dot{\phi}\right)^2 + \left(\dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta} + l\cos(\phi)\dot{\phi}\right)^2\right]\\ &= \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2\dot{r}\dot{\phi}l\left(\sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi)\right) + 2rl\dot{\theta}\dot{\phi}\left(\cos(\theta)\cos(\phi)\right.\right.\\ &= \frac{m}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2\dot{r}\dot{\phi}l\sin(\theta-\phi) + 2rl\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right].\end{aligned}$$

So the total KE is

$$m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \frac{l^2\dot{\phi}^2}{2} + \dot{r}\dot{\phi}l\sin(\theta-\phi) + rl\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right].$$

.