Lagrangian Mechanics - Generalised Coordinates

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1. The equations of motion are

$$\begin{aligned}\frac{d}{dt}(\dot{q}_1) - q_1 &= 0 \\ \frac{d}{dt}(\dot{q}_2) - q_2 &= 0\end{aligned}$$

so we calculate

$$\frac{d}{dt}(\dot{q}_1 q_2 - \dot{q}_2 q_1) = \ddot{q}_1 q_2 + \dot{q}_1 \dot{q}_2 - \ddot{q}_2 q_1 - \dot{q}_1 \dot{q}_2 = q_1 q_2 - q_2 q_1 = 0$$

using the equations of motion to get rid of the second derivative terms.

2. We have

$$\begin{aligned}\dot{x} &= \dot{r}\sin(\theta)\cos(\varphi) + r\cos(\theta)\cos(\varphi)\dot{\theta} - r\sin(\theta)\sin(\varphi)\dot{\varphi} \\ \dot{y} &= \dot{r}\sin(\theta)\sin(\varphi) + r\cos(\theta)\sin(\varphi)\dot{\theta} + r\sin(\theta)\cos(\varphi)\dot{\varphi} \\ \dot{z} &= \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta}\,.\end{aligned}$$

Putting these into

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

leads to the desired result after a little bit of straightforward algebra.

3. The kinetic energy is as in the previous problem, setting $r = L$ (which is a constant, so $\dot{r} = 0$), and the potential energy is $mgz = mgL\cos(\theta)$. This implies that

$$L = \frac{1}{2}m(L^2\dot{\theta}^2 + L^2\sin^2(\theta)\dot{\varphi}^2) - mgL\cos(\theta)\,.$$

Accordingly, the equations of motion are

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= mL^2\ddot{\theta} - mL^2\sin(\theta)\cos(\theta)\dot{\varphi}^2 - mgL\sin(\theta) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} &= \frac{d}{dt}\left(mL^2\sin^2(\theta)\dot{\varphi}\right) = mL^2(2\sin(\theta)\cos(\theta)\dot{\theta}\dot{\varphi} + \sin^2(\theta)\ddot{\varphi}) = 0\,.\end{aligned}$$