Lagrangian Mechanics
Symmetries and conservation laws I

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1. (a) The Lagrangian doesn't depend on θ so conserved quantity is

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{2z^2\dot{\theta}}{w^2}.$$

(b) The Lagrangian doesn't depend on ϕ so conserved quantity is

$$\frac{\partial L}{\partial \dot{\phi}} = r^2 \sin^2(\theta)\dot{\phi}.$$

(c) The Lagrangian doesn't depend on x, y or z, so the conserved quantities are

$$p_x = \frac{\partial L}{\partial \dot{x}} = -\frac{\dot{x}}{2\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}}$$
$$p_y = -\frac{\dot{y}}{2\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}}$$
$$p_z = -\frac{\dot{z}}{2\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}}.$$

2. (a) Under the transformation

$$\begin{aligned} q_2 &\rightarrow q_2' = q_2 + \epsilon q_3 \\ q_3 &\rightarrow q_3' = q_3 - \epsilon q_2 \end{aligned}$$

the quantity $q_2^2 + q_3^2$ transforms as

$$q_2^2 + q_3^2 \rightarrow (q_2 + \epsilon q_3)^2 + (q_3 - \epsilon q_2)^2 = q_2^2 + q_3^2 + O(\epsilon^2).$$

A similar calculation (with dots on) shows that $\dot{q}_2^2 + \dot{q}_3^2$ only changes by terms of $O(\epsilon^2)$. Thus to this order the Lagrangian is invariant and the transformation is a symmetry with $F = 0$. By definition we have $a_2 = q_3$ and $a_3 = -q_2$, (and $a_1 = 0$) so Noether's theorem tells us that

$$Q = a_i \frac{\partial L}{\partial \dot{q}_i} - F = q_3 \frac{\partial L}{\partial \dot{q}_2} - q_2 \frac{\partial L}{\partial \dot{q}_3} - 0 = q_3 q_1^2 \dot{q}_2 - q_2 q_1^2 \dot{q}_3$$

is conserved.

(b) Expanding the derivative, we have

$$\begin{aligned}\frac{dQ}{dt} &= q_3\frac{d(q_1^2\dot q_2)}{dt} + q_1^2(\dot q_3\dot q_2) - q_2\frac{d(q_1^2\dot q_3)}{dt} - q_1^2(\dot q_2\dot q_3)\\ &= q_3\frac{d(q_1^2\dot q_2)}{dt} - q_2\frac{d(q_1^2\dot q_3)}{dt}\,.\end{aligned}$$

Meanwhile, the Euler-Lagrange equations for q_2 and q_3 are

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_2}\right) - \frac{\partial L}{\partial q_2} &= \frac{d(q_1^2\dot q_2)}{dt} + q_2 = 0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_3}\right) - \frac{\partial L}{\partial q_3} &= \frac{d(q_1^2\dot q_3)}{dt} + q_3 = 0\end{aligned}$$

so $\dot Q = 0$ now follows by substitution.

3. (a) Under the transformation $x \to (1+\epsilon)x$ and $\dot x \to (1+\epsilon)\dot x$. So $\frac{\dot x}{x}$ is invariant, and the only term in the Lagrangian which changes is the term containing $\ln(x)$. Under the transformation

$$\begin{aligned}\delta L &= L(x',y',\dot x',\dot y') - L(x,y,\dot x,\dot y) = (\ln(x') - \ln(x))\,\frac{\dot y}{y}\\ &= (\ln(x(1+\epsilon)) - \ln(x))\,\frac{\dot y}{y} = \ln(1+\epsilon)\frac{d}{dt}\,(\ln(y))\\ &= \epsilon\frac{d}{dt}\,(\ln(y)) + O(\epsilon^2)\end{aligned}$$

so we can take $F = \ln(y)$. By Noether's theorem we have that

$$Q_x = x\frac{\partial L}{\partial \dot x} - F = \left(\frac{2\dot x}{x} - \ln(y)\right) - \ln(y) = \frac{2\dot x}{x} - 2\ln(y)$$

is conserved.

(b) Considering the transformation $x \to x$, $y \to (1+\epsilon)y$ we similarly find that $\delta L = -\epsilon\frac{d}{dt}\ln(x) + O(\epsilon^2)$ so $F = -\ln(x)$. The corresponding conserved quantity is

$$Q_y = \frac{2\dot y}{y} + 2\ln(x)$$

(c) From the conservation of Q_x an Q_y we can write that

$$\begin{aligned}\frac{Q_x}{2} &= \frac{d}{dt}(\ln(x)) - \ln(y) = C_1 = 0\\ \frac{Q_y}{2} &= \frac{d}{dt}(\ln(y)) + \ln(x) = C_2 = 1 + \ln(2)\end{aligned}$$

Substituting $\ln(y)$ from the first equation we find

$$\frac{d^2}{dt^2}(\ln(x)) + \ln(x) = C_2 = 1 + \ln(2)$$

which has the solution $\ln(x) = \alpha\cos(t) + \beta\sin(t) + 1 + \ln(2)$. Using the initial conditions to find α, β we find that

$$\begin{aligned} \ln(x) &= 1 - \cos(t) + \ln(2) \\ \ln(y) &= \frac{d}{dt}(\ln(x)) = \sin(t). \end{aligned}$$

4. (a) Using that

$$\begin{aligned} \dot{w} &\to \cosh(\theta)\dot{w} + \sinh(\theta)\dot{x} \\ \dot{x} &\to \cosh(\theta)\dot{x} + \sinh(\theta)\dot{w} \end{aligned}$$

we find that

$$\dot{w}^2 - \dot{x}^2 \to (\cosh(\theta)\dot{w} + \sinh(\theta)\dot{x})^2 - (\cosh(\theta)\dot{x} + \sinh(\theta)\dot{w})^2 = \dot{w}^2 - \dot{x}^2.$$

As $\dot{y}$ and $\dot{z}$ are not altered by the transformation, it is clear that L is invariant.

(b) To order $O(\epsilon)$, we have $\cosh(\epsilon) = 1$ and $\sinh(\epsilon) = \epsilon$. Therefore the infinitesimal form of the transformation is just

$$\begin{aligned} w &\to w + \epsilon x \\ x &\to x + \epsilon w. \end{aligned}$$

As L is invariant under this transformation, Noethers theorem tells us that the charge Q is conserved where

$$Q = x\frac{\partial L}{\partial \dot{w}} + w\frac{\partial L}{\partial \dot{x}} = \frac{m(w\dot{x} - x\dot{w})}{\sqrt{\dot{w}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}}.$$

5. The first bit is standard bookwork; if $L = L(q_i, \dot{q}_i)$ then

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) \\ \Rightarrow \frac{d}{dt}\left(\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} - L\right) &= 0. \end{aligned}$$

Applying this to the given Lagrangian we find

$$\begin{aligned} E &= \dot{u}\frac{\partial L}{\partial \dot{u}} + \dot{v}\frac{\partial L}{\partial \dot{v}} - L \\ &= \dot{u}^2 + \dot{v}\frac{\dot{v}}{2u^2} - \frac{1}{2}\dot{u}^2 - \frac{1}{2u^2}\dot{v}^2 + \frac{1}{2u^2} \\ &= \frac{1}{2}\dot{u}^2 + \frac{1}{2}\frac{\dot{v}^2}{u^2} + \frac{1}{2u^2}, \end{aligned}$$

As v is an ignorable coordinate we have that $\frac{\partial L}{\partial \dot{v}} = \frac{\dot{v}}{u^2}$ is also conserved.

6. (a)

$$E = \frac{1}{w^2}\left(\dot{w}^2 + z^2\dot{\theta}^2\right) - \frac{1}{2}\dot{z}^2 - \sqrt{z^2+w^2}.$$

(b)

$$E = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\phi}^2\right) + mr\cos(\theta).$$

(c)

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}} - L.$$

We find that

$$\dot{x}\frac{\partial L}{\partial \dot{x}} = -\frac{\dot{x}^2}{2\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}},$$

so substituting this is in to E, together with the similar expressions for y and z gives

$$E = -\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{2\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}} - \frac{1}{2}\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2} = -\frac{1}{2\sqrt{1-\dot{x}^2-\dot{y}^2-\dot{z}^2}}.$$