Lagrangian Mechanics
Symmetries and conservation laws II

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1. (a) The Euler-Lagrange equation for z is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = m\ddot{z} + mg - f(t) = 0\,.$$

Note that this is the only Euler-Lagrange equation: there is no Euler-Lagrange equation for t (which gets treated separately in the Lagrangian formalism, it is not a generalized coordinate).

(b) The energy is

$$E = \dot{z}\frac{\partial L}{\partial \dot{z}} - L = \frac{1}{2}m\dot{z}^2 + mgz - zf(t)\,.$$

(c) Taking the time derivative we have

$$\begin{aligned}\frac{dE}{dt} &= m\dot{z}\ddot{z} + mg\dot{z} - \dot{z}f(t) - z\frac{df}{dt} \\ &= \dot{z}\,\underbrace{(\ddot{z} + mg - f(t))}_{=0 \text{ due to E-L equations}} - z\frac{df}{dt}\,.\end{aligned}$$

On the other hand:

$$\frac{\partial L}{\partial t} = z\frac{df}{dt}$$

since z and t are treated as independent variables when taking partial derivatives, and $f(t)$ is a function of t only, so

$$\frac{\partial f(t)}{\partial t} = \frac{df(t)}{dt}\,,$$

by definition of partial derivative.

2. To first order in the rotation parameter ϵ, rotations act as

$$\begin{aligned}x &\to x' = x - \epsilon y \\ y &\to y' = y + \epsilon x\,.\end{aligned}$$

Under this transformation, the Lagrangian becomes (again to first order in ϵ)

$$L \to L' = L(x', y') = L + \epsilon(2axy - 2bxy - 3cy^2x)\,.$$

For later convenience, let me introduce $K(x, y; a, b, c) := 2(a - b)xy - 3cy^2x$, so that

$$L' = L + \epsilon K\,.$$

This transformation will be a symmetry if there some $F(x, y, t)$ such that

$$L' = L + \epsilon \frac{dF}{dt} + \mathcal{O}(\epsilon^2)$$

or in other words if some $F(x, y, t)$ exist such that

$$K(x, y; a, b, c) = \frac{dF(x, y, t)}{dt} .$$

By the chain rule

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial y}\dot{y} + \frac{\partial F}{\partial t} .$$

Since K includes no factors of $\dot{x}$ or $\dot{y}$, we have

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$$

or in other words F can only depend on time, $F(t)$.

Now we notice that the only way that we could have $K(x, y; a, b, c) = \frac{dF(t)}{dt}$ is if both sides are constant, since the two sides of the equation depend on different sets of variables. So the problem reduces to choosing values for a, b, c such that K is a constant. Clearly, the only solution to this is $a = b$ and $c = 0$, which implies $K = 0$ and F constant.

3. Rotations in polar coordinates r, θ are generated by

$$r \to r \qquad ; \qquad \theta \to \theta + \epsilon \qquad ; \qquad \dot{r} \to \dot{r} \qquad ; \qquad \dot{\theta} \to \dot{\theta} .$$

Under this transformation we have

$$\begin{aligned} L \to L' &= L(r, \theta + \epsilon, \dot{r}, \dot{\theta}) \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r, \theta + \epsilon) . \end{aligned}$$

To first order in ϵ, this is

$$L' = L - \epsilon \frac{\partial V}{\partial \theta} .$$

In order for this to be a symmetry, we need to find an $F(r, \theta, t)$ such that

$$L' = L + \epsilon \frac{dF}{dt}$$

to first order in ϵ. As in the previous problem, since there are no velocities in $\frac{\partial V}{\partial \theta}$, we conclude that F is a function of t only, and we need to solve:

$$\frac{\partial V(r, \theta)}{\partial \theta} = -\frac{dF(t)}{dt} .$$

The two sides of this equation depend on different sets of coordinates, so they can only agree if they are both equal to some constant, which I will call k. We need to solve

$$\begin{aligned}\frac{\partial V(r,\theta)}{\partial \theta} &= k \\ \frac{dF(t)}{dt} &= -k\,.\end{aligned}$$

Integrating these equations, we find

$$\begin{aligned}V(r,\theta) &= k\theta + P(r) \\ F &= -kt + d\end{aligned}$$

for d an arbitrary constant and $P(r)$ an arbitrary function of r.

If r and θ were arbitrary generalised coordinates this would be the end of the story. For polar coordinates, we might want to impose a further condition coming from the fact that θ is periodic, namely (r,θ) and $(r,\theta+2\pi)$ denote exactly the same point in configuration space $\mathcal{C}$. If we impose that the Lagrangian is a well defined function on $\mathcal{C}$, and not well defined simply up to a constant, then we obtain the further constraint $k = 0$, and the only acceptable potential is of the form $P(r)$. Whether we impose this constraint depends on whether we want the Lagrangian to be well defined as a function on $\mathcal{C}$, or only well defined up to a constant. This is a fairly subtle issue, so for the purposes of this homework both possibilities (setting $k = 0$ or leaving it arbitrary) are acceptable. In particular it is acceptable to set $k = 0$ from the beginning, and conclude that V is a function of r only.