Lagrangian Mechanics
Normal modes

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1. The equations of motion for the given Lagrangian are

$$\begin{aligned} \ddot{x} + \frac{2x}{1-x^2} + \sin(x+y) &= 0 \\ \ddot{y} + \sin(x+y) &= 0 \end{aligned}$$

which are solved by $x = y = 0$. Recalling that $\ln(1+\epsilon) = \epsilon + O(\epsilon^2)$ we can write the Lagrangian as

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - x^2 + \left(1 - \frac{(x+y)^2}{2}\right) + O(|(x,y)|^3).$$

so that

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{3}{2}x^2 - xy - \frac{1}{2}y^2$$

where we have also ditched an irrelevant constant. The equation following from this approximate Lagrangian are

$$\begin{aligned} \ddot{x} + 3x + y &= 0 \\ \ddot{y} + x + y &= 0. \end{aligned}$$

The equations of motion are in the form $\ddot{q}_i + A_{ij}q_j$ where the matrix A is given by

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

The normal modes are of the form $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{v}e^{i\omega t}$ where $\mathbf{v}$ is an eigenvector of A and ω^2 is the corresponding eigenvalue. In this case the characteristic equation is $\lambda^2 - 4\lambda + 2 = 0$ which has solutions $\lambda = 2 \pm \sqrt{2}$ and corresponding to these eigenvalues we have eigenvectors $(1, \pm\sqrt{2} - 1)$. So the normal modes are

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} e^{\pm it\sqrt{\sqrt{2}+2}}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{2} - 1 \end{pmatrix} e^{\pm it\sqrt{-\sqrt{2}+2}}$$

The (real) general solution can be written

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ \sqrt{2} - 1 \end{pmatrix} \left(a_1 \cos(t\sqrt{\sqrt{2}+2}) + b_1 \sin(t\sqrt{\sqrt{2}+2})\right) \\ &+ \begin{pmatrix} 1 \\ -\sqrt{2} - 1 \end{pmatrix} \left(a_2 \cos(t\sqrt{-\sqrt{2}+2}) + b_2 \sin(t\sqrt{-\sqrt{2}+2})\right). \end{aligned}$$

The vanishing of x and y at $t = 0$ tells us that $a_1 = a_2 = 0$, whilst the conditions on the velocity at $t = 0$ give that

$$\begin{pmatrix} 0.1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} b_1 \sqrt{\sqrt{2}+2} + \begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix} b_2 \sqrt{-\sqrt{2}+2}$$

which we can solve to find (I give a numerical approximation to the solution, since the analytic form is not particularly nice)

$$\begin{aligned} b_1 &\approx 0.046194 \\ b_2 &\approx 0.019134. \end{aligned}$$

2. (a) The stationary points of the potential are located at the points where

$$\vec{\nabla} V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y} \right) = 0\,.$$

In our case we have

$$\vec{\nabla} V = (2x, 4(y+1)(y-1)y)$$

which has solutions at $x = 0$, $y \in \{-1, 0, 1\}$.

(b) We have

$$\begin{aligned} \mathsf{A}_{11} &= \frac{\partial^2 V}{\partial x \partial x} = 2 \\ \mathsf{A}_{12} = \mathsf{A}_{21} &= \frac{\partial^2 V}{\partial x \partial y} = 0 \\ \mathsf{A}_{22} &= \frac{\partial^2 V}{\partial y \partial y} = 4(3y^2 - 1) \end{aligned}$$

or in matrix form

$$\mathsf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 4(3y^2-1) \end{pmatrix}$$

(c) The stationary points at $(0, \pm 1)$ have all positive eigenvalues, so they are minima, while the one at $(0, 0)$ has one positive and one negative eigenvalue, so it is a saddle point.

3. (a) The Lagrangian is

$$L = \underbrace{\frac{1}{2} m \dot{y}^2}_{T} - \underbrace{\left(\frac{1}{2} \kappa (y+a)^2 + mgy \right)}_{V}$$

so the Euler-Lagrange equation is

$$m\ddot{y} + \kappa y + a\kappa + mg = 0\,.$$

(b) The solution that does not depend on time is $y = -a - mg/\kappa$. This is a bit lower than the natural length of the spring, which is reasonable since gravity is pulling toward negative values of y.

(c) This is true, since in the previous point the contribution from the kinetic term cancelled (since we are looking to equilibrium position). So the non-trivial contribution to the Euler-Lagrange equation for the equilibrium solution was precisely

$$-\frac{\partial L}{\partial y} = \frac{\partial V}{\partial y}\,.$$

(d) We can introduce $q = y + a + mg/\kappa$. Then

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\kappa q^2 + \left(amg + \frac{mg^2}{2\kappa}\right)\,.$$

(e) The Euler-Lagrange equation is

$$m\ddot{q} + \kappa q = 0\,.$$

(f) We can convert this into canonical form dividing by m. Then

$$\ddot{q} + \frac{\kappa}{m} q = 0$$

with solution

$$q = \alpha\cos(\omega t) + \beta\sin(\omega t)$$

with $\omega^2 = \frac{\kappa}{m}$.

4. The Euler-Lagrange equations are

$$m\ddot{q}_1 + \kappa q_1 - \kappa q_2 = 0\,,$$
$$M\ddot{q}_2 - \kappa q_1 + \kappa q_2 = 0\,.$$

Dividing the first equation by m and the second by M we equivalently have

$$\ddot{q}_1 + \zeta q_1 - \zeta q_2 = 0\,,$$
$$\ddot{q}_2 - \eta q_1 + \eta q_2 = 0$$

with $\zeta \coloneqq \kappa/m$ and $\eta \coloneqq \kappa/M$. In matrix form, this is

$$\ddot{\vec{q}} + \mathsf{A}\vec{q} = 0$$

with

$$\mathsf{A} \coloneqq \begin{pmatrix} \zeta & -\zeta \\ -\eta & \eta \end{pmatrix}\,.$$

This matrix was eigenvalues $\lambda^{(1)} = 0$ and $\lambda^{(2)} = \zeta+\eta$, with corresponding eigenvectors

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \vec{v}^{(2)} = \begin{pmatrix} \zeta \\ -\eta \end{pmatrix}$$

and the general solution is

$$\vec{q}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (Ct + D) + \begin{pmatrix} \zeta \\ -\eta \end{pmatrix} \left(\alpha \cos(\sqrt{\zeta+\eta}t) + \beta \sin(\sqrt{\zeta+\eta}t)\right) .$$

5. (a) The kinetic term is

$$\begin{aligned} T &= \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \\ &= \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) \end{aligned}$$

and the potential energy is

$$\begin{aligned} V &= \frac{1}{2}\kappa(x_2 - x_1 - a)^2 + \frac{1}{2}\kappa(x_3 - x_2 - a)^2 \\ &= \frac{1}{2}\kappa(q_2 - q_1)^2 + \frac{1}{2}\kappa(q_3 - q_2)^2 . \end{aligned}$$

The Lagrangian is $L = T - V$ and the corresponding Euler-Lagrange equations are (in vector form)

$$\ddot{\vec{q}} + \mathsf{A}\vec{q} = 0$$

with

$$\mathsf{A} := \begin{pmatrix} \kappa & -\kappa & 0 \\ -\kappa & 2\kappa & -\kappa \\ 0 & -\kappa & \kappa \end{pmatrix} .$$

(b) The eigenvalues are $\lambda^{(i)} = \{0, \kappa, 3\kappa\}$ with corresponding eigenvectors:

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad ; \quad \vec{v}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad ; \quad \vec{v}^{(3)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The first one is a zero mode, the two others are normal modes with positive frequency.

(c) The zero mode $\lambda^{(1)}$ corresponds to translating the whole system, keeping the relative distances constant. The $\lambda^{(2)}$ mode corresponds to pulling apart the two particles at the left and right by the same amount in opposite directions, keeping the central particle fixed. The $\lambda^{(3)}$ mode corresponds to the left and right particles moving in one direction, while the central one moves in the opposite direction, keeping the centre of mass fixed.

The three eigenvectors are linearly independent, and the given initial conditions are proportional to $\vec{v}^{(2)}$ starting from rest. So we will only excite this normal mode, and the subsequent motion will be described by

$$q(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \delta \cos(t\sqrt{\kappa}) = \begin{pmatrix} \delta \\ 0 \\ -\delta \end{pmatrix} \cos(t\sqrt{\kappa})\,.$$

6. The Lagrangian for the particle is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz.$$

Here $z = ax^2 + by^2 + 2hxy$, so in terms of x, y the Lagrangian is

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{m}{2}\left(2ax\dot{x} + 2by\dot{y} + 2hx\dot{y} + 2h\dot{x}y\right)^2 - mg\left(ax^2 + by^2 + 2hxy\right).$$

Now recall that we are only interested in normal modes, which rely on approximating L by terms which are up to quadratic in the 'small displacement' from equilibrium. Here we assume that the bottom of the bowl is at $x = y = 0$, so if we look at terms which are at most quadratic in small things $(x, y, \dot{x}, \dot{y})$ we see that we can drop the foul term coming from $\dot{z}^2$ and we are left with the approximate Lagrangian

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mg\left(ax^2 + by^2 + 2hxy\right).$$

Its straight forward to show that the equations of motions are

$$\begin{aligned} \ddot{x} + 2gax + 2ghy &= 0 \\ \ddot{y} + 2ghx + 2gby &= 0. \end{aligned}$$

So we know that the angular frequency ω of the normal modes are given by $\omega^2 = \lambda$ where λ is an eigenvalue of the matrix

$$A = 2g\begin{pmatrix} a & h \\ h & b \end{pmatrix}.$$

The eigenvalues of $\begin{pmatrix} a & h \\ h & b \end{pmatrix}$ are given by the solutions to the characteristic equations $\lambda^2 - (a+b)\lambda + ab - h^2 = 0$. So using the normal formula the eigenvalues of A are

$$\lambda_\pm = g\left(a + b \pm \sqrt{(a-b)^2 + 4h^2}\right).$$

So we can define angular frequencies of the normal modes to be $\omega_\pm = \sqrt{\lambda_\pm}$ and if you want to be really pedantic, the periods of the normal modes will be $2\pi/\omega_\pm$. Now we imagine that the particle is constrained to move along some vertical plane $y = kx$, but sticking on the surface of the bowl (i.e. constrained to move on some vertical cross-section of the bowl). The questions asks you to show that the maximum and minimum periods of these motions (as I vary k) coincide with the periods of the normal modes. Actually this is no big surprise. Our bowl is given by the quadratic surface $z = ax^2 + by^2 + 2hxy \propto A_{ij}x_ix_j$, where $x_1 = x$, $x_2 = y$. Surfaces of constant z are ellipses, and the minor and major axes of this ellipse coincide with the eigenvectors of the matrix A. So the normal modes, which are in the direction of the eigenvector, will lie along these directions which are the steepest and shallowest cross-sections of the bowl (as the major and minor axes are the longest and shortest distances across the ellipse). Now to show this rather more mathematically. Along the line $y = kx$ are approximate Lagrangian becomes

$$L = \frac{m(1+k^2)\dot{x}^2}{2} - mg(a + bk^2 + 2hk)x^2,$$

and the corresponding equation of motion is $\ddot{x}(1+k^2) + 2g(a + bk^2 + 2hk)x = 0$. So the square of the angular frequency is simply

$$\lambda_k = \frac{2g(a + bk^2 + 2hk)}{1+k^2}.$$

We would like to show that the max/min of λ_k coincides with $\lambda_\pm$. At such values we must have

$$\frac{d\lambda_k}{dk} = 0 = -\frac{4gk(a + bk^2 + 2hk)}{(1+k^2)^2} + \frac{2g\,(2bk + 2h)}{(1+k^2)}.$$

Rearranging this gives $-hk^2 + (b-a)k + h = 0$, so

$$k_\pm = \frac{b - a \pm \sqrt{(a-b)^2 + 4h^2}}{2h}.$$

We can substitute this back into our expression for λ_k, and after quite a lot of gymnastics we will recover $\lambda_k = \lambda_\pm$ as desired. But its quicker to rearrange the equation for λ_k to give

$$k^2(2gb - \lambda_k) + 4ghk + (2ga - \lambda_k) = 0.$$

If we fix λ_k this can gives an equation for k. If you think about it for a second, you will see that at the maximum/minimum of λ_k the two roots for k must coincide so the usual '$b^2 - 4ac$' in the quadratic formula must vanish. This tells us that at the max/min values, λ_k must satisfy

$$4\,(\lambda_k - 2gb)\,(\lambda_k - 2ga) - (4gh)^2 \;=\; 0,$$

which is precisely the quadratic equation (characteristic polynomial of the matrix A) which $\lambda_\pm$ satisfies.

7. If I label the angles from the (downward) vertical to be θ_1 and θ_2 for the pendula with bobs of masses M and m respectively, then the KE of the system is

$$\frac{l^2}{2}\left(M\dot{\theta}_1^2 + m\dot{\theta}_2^2\right),$$

the gravitation potential energy is $-gl\left(M\cos(\theta_1) + m\cos(\theta_2)\right)$ which up to a constant is appromiately $gl\left(M\theta_1^2 + m\theta_2^2\right)/2$ and the potential energy of the spring is approximately $kl^2\left(\theta_2 - \theta_1\right)^2/2$. Putting this together we get an approximate Lagrangian

$$L = \frac{l^2}{2}\left(M\dot{\theta}_1^2 + m\dot{\theta}_2^2\right) - \frac{gl}{2}\left(M\theta_1^2 + m\theta_2^2\right) - \frac{kl^2}{2}\left(\theta_2 - \theta_1\right)^2.$$

The equations of motion which follow from this can be put in the form $\ddot{\theta}_i + A_{ij}\theta_j = 0$, where

$$A = \begin{pmatrix} \frac{g}{l} + \frac{k}{M} & -\frac{k}{M} \\ -\frac{k}{m} & \frac{g}{l} + \frac{k}{m} \end{pmatrix}.$$

The eigenvalues are given by the characteristic polynomial

$$\lambda^2 - \left(\frac{2g}{l} + \frac{k}{M} + \frac{k}{m}\right)\lambda + \frac{g^2}{l^2} + \frac{g}{l}\left(\frac{k}{M} + \frac{k}{m}\right) = 0.$$

If we call $k/M + k/m = 2\kappa$, then we find that

$$\lambda_\pm = \frac{g}{l} + 2\kappa, \frac{g}{l},$$

with corresponding eigenvalues

$$\underline{a}_\pm = \begin{pmatrix} m \\ -M \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus the general form of the solution is

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} m \\ -M \end{pmatrix}\left(\alpha_1\sin(\omega_+ t) + \beta_1\cos(\omega_+ t)\right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(\alpha_2\sin(\omega_- t) + \beta_2\cos(\omega_- t)\right),$$

where $\omega_+^2 = \lambda_+$, $\omega_-^2 = \lambda_-$. As the pendula are initially at rest we immediately deduce $\alpha_1 = \alpha_2 = 0$. At $t = 0$ we have

$$\begin{pmatrix} \frac{a}{l} \\ 0 \end{pmatrix} = \begin{pmatrix} m \\ -M \end{pmatrix}\beta_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\beta_2,$$

which gives

$$\beta_1 = \frac{a}{l\left(M + m\right)}, \beta_2 = \frac{Ma}{l\left(M + m\right)}.$$

This gives as the particular form of the solution

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} m \\ -M \end{pmatrix} \left(\frac{a}{l\,(M+m)} \cos(\omega_+ t) \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(\frac{Ma}{l\,(M+m)} \cos(\omega_- t) \right).$$

From this we can see that

$$|\theta_2| = \left| \frac{Ma}{l\,(M+m)} \left(\cos(\omega_- t) - \cos(\omega_+ t) \right) \right| \leq \frac{2Ma}{l\,(M+m)}$$

and given that the displacement is approximately $l\theta_2$ the result follows.

8. (a) We have $x_1 = q_1$ and $x_2 = q_2 + a$. The Lagrangian in the x-coordinates is

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}\kappa((x_2 - x_1) - a)^2$$

and the result follows immediately from substitution.

(b) We use the technology of normal modes. The Euler-Lagrange equations for the system are

$$\ddot{q}_1 + \kappa q_1 - \kappa q_2 = 0$$
$$\ddot{q}_2 - \kappa q_1 + \kappa q_2 = 0$$

or in matrix form $\ddot{\vec{q}} + \mathsf{A}\vec{q} = 0$ with

$$\mathsf{A} = \begin{pmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{pmatrix}.$$

This matrix has eigenvalues $\lambda^{(1)} = 0$ and $\lambda^{(2)} = 2\kappa$, with associated eigenvectors

$$\vec{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \vec{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

respectively. The general solution is then

$$\vec{q}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (Ct + D) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (\alpha \cos(\omega t) + \beta \sin(\omega t))$$

with $\omega = \sqrt{2\kappa}$ and C, D, α, β constants to be determined.

(c) From $(q_1, q_2) = (0, 0)$ at $t = 0$ we find

$$\begin{pmatrix} D + \alpha \\ D - \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $D = \alpha = 0$. From $(\dot{q}_1, \dot{q}_2) = (v, v)$ we then find:

$$\begin{pmatrix} C + \beta\omega \\ C - \beta\omega \end{pmatrix} = \begin{pmatrix} v \\ v \end{pmatrix}$$

so $\beta = 0$ and $C = v$. We thus find as our solution

$$\vec{q}(t) = \begin{pmatrix} vt \\ vt \end{pmatrix}$$

which encodes constant velocity motion of the whole system, without oscillation.

(d) We now start from rest. Using the fact that $\dot{\vec{q}} = 0$ for $t = 0$, we find

$$\begin{pmatrix} C + \omega\beta \\ C - \omega\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies $C = \beta = 0$. From the initial position at $t = 0$ we find

$$\begin{pmatrix} D + \alpha \\ D - \alpha \end{pmatrix} = \begin{pmatrix} c \\ -c \end{pmatrix}$$

which implies $D = 0$, $\alpha = c$. The solution is then

$$\vec{q}(t) = \begin{pmatrix} c \\ -c \end{pmatrix} \cos(\omega t) \, .$$

9. (a) In the y coordinates we have

$$L = \frac{1}{2}(\dot{y}_1^2 + \dot{y}_2^2) - \frac{1}{2}\kappa(y_1 + a)^2 - \frac{1}{2}\kappa(y_2 - y_1 + a)^2 - gy_1 - gy_2 \, .$$

Note that our conventions are that y increases upwards, so the unextended springs will have negative values for y_1 and y_2. Introducing q_1 and q_2 such that $y_1 = q_1 - a$, $y_2 = q_2 - 2a$ we find

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}\kappa q_1^2 - \frac{1}{2}\kappa(q_2 - q_1)^2 - gq_1 - gq_2 + 3ga \, .$$

The last term is a constant, so it will not affect the equations of motion.

(b) The equations of motion for the Lagrangian above are

$$\begin{aligned} \ddot{q}_1 + 2\kappa q_1 - \kappa q_2 + g &= 0 \\ \ddot{q}_2 - \kappa q_1 + \kappa q_2 + g &= 0 \, . \end{aligned}$$

The position of equilibrium is the solution of these equations with q_1 and q_2 constant in time (which implies, in particular, that $\ddot{q}_1 = \ddot{q}_2 = 0$.) That is, we need to solve

$$\begin{aligned} 2\kappa q_1^{(0)} - \kappa q_2^{(0)} + g &= 0 \\ -\kappa q_1^{(0)} + \kappa q_2 + g &= 0 \, . \end{aligned}$$

In matrix form this is

$$\mathsf{A}\vec{q}^{(0)} = -g\begin{pmatrix}1\\1\end{pmatrix}$$

with

$$\mathsf{A} = \begin{pmatrix}2\kappa & -\kappa\\ -\kappa & \kappa\end{pmatrix}.$$

We have

$$\mathsf{A}^{-1} = \frac{1}{\kappa}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$$

so

$$\begin{pmatrix}q_1^{(0)}\\ q_2^{(0)}\end{pmatrix} = -g\mathsf{A}^{-1}\begin{pmatrix}1\\1\end{pmatrix} = -\frac{g}{\kappa}\begin{pmatrix}2\\3\end{pmatrix}$$

(c) Introducing $r_1 = q_1 + \frac{2g}{\kappa}$, $r_2 = q_2 + \frac{3g}{\kappa}$, we find a new Lagrangian

$$L = \frac{1}{2}(\dot{r}_1^2 + \dot{r}_2^2) - \frac{1}{2}\kappa(2r_1^2 - 2r_1r_2 + r_2^2) + \frac{5g^2}{2\kappa} + 3ga\,.$$

As a small check, note that there are no longer any linear terms in the new variables, as should be the case since we should now be expanding around a minimum.

In these new variables the Euler-Lagrange equations are

$$\ddot{r}_1 + 2\kappa r_1 - \kappa r_2 = 0$$
$$\ddot{r}_2 - \kappa r_1 + \kappa r_2 = 0$$

or in vector notation

$$\ddot{\vec{r}} + \mathsf{A}\vec{r} = 0$$

with

$$\mathsf{A} = \begin{pmatrix}2\kappa & -\kappa\\ -\kappa & \kappa\end{pmatrix}$$

The eigenvalue equation is

$$\det(\lambda - \mathsf{A}) = \lambda^2 - 3\lambda + 1 = 0$$

with solutions

$$\lambda^{(1)} = \kappa\frac{3+\sqrt{5}}{2} \qquad ; \qquad \lambda^{(2)} = \kappa\frac{3-\sqrt{5}}{2}\,.$$

The corresponding eigenvectors are

$$\vec{v}^{(1)} = \begin{pmatrix}1\\ \frac{1-\sqrt{5}}{2}\end{pmatrix} \qquad ; \qquad \vec{v}^{(2)} = \begin{pmatrix}1\\ \frac{1+\sqrt{5}}{2}\end{pmatrix}.$$

The general solutions is, as usual:

$$\vec{r}(t) = \sum_{i\in\{1,2\}} \vec{v}^{(i)}\left(\alpha^{(i)}\cos(\omega^{(i)}t) + \beta^{(i)}\sin(\omega^{(i)}t)\right)$$

with $\omega^{(i)} = \sqrt{\lambda^{(i)}}$.