Week 7 problems

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1. (a) The Euler-Lagrange equations are

$$\ddot{q}_1 + 5\sin(q_1) - 2q_2\cos(q_1 q_2) = 0$$
$$\ddot{q}_2 - 2q_1\cos(q_1 q_2) + 2\sin(q_2) = 0\,.$$

Recalling that $q_1(t) = q_2(t) = 0$ implies $\ddot{q}_1(t) = \ddot{q}_2(t) = 0$, it is clear that $q_1(t) = q_2(t) = 0$ solves these equations.

(b) For small displacements, to second order in the q_i, we have $\cos(q_i) = 1 - \frac{1}{2}q_i^2 + \ldots$, $\sin(q_1 q_2) = q_1 q_2 + \ldots$. Plugging these expansions into the original Lagrangian we find

$$L_{\text{approx}} = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{5}{2}q_1^2 + 2q_1 q_2 - q_2^2 + 7\,.$$

The constant at the end does not influence the equations of motion, so it can be ignored in what follows.

(c) The Euler-Lagrange equations are

$$\ddot{q}_1 + 5q_1 - 2q_2 = 0$$
$$\ddot{q}_2 - 2q_1 + 2q_2 = 0\,.$$

In matrix form:

$$\ddot{\mathbf{q}} + \mathsf{A}\mathbf{q} = 0$$

with

$$\mathsf{A} = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}.$$

This matrix has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 6$, and corresponding eigenvectors

$$\mathbf{v}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad ; \qquad \mathbf{v}^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Applying the general normal modes discussion, we have

$$\mathbf{q}(t) = \mathbf{v}^{(1)}(\alpha^{(1)}\cos(t) + \beta^{(1)}\sin(t)) + \mathbf{v}^{(2)}(\alpha^{(2)}\cos(t\sqrt{6}) + \beta^{(2)}\sin(t\sqrt{6}))\,.$$

(d) The fact that we start from rest implies that

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{v}^{(1)}\beta^{(1)} + \sqrt{6}\mathbf{v}^{(2)}\beta^{(2)}$$

which implies $\beta^{(1)} = \beta^{(2)} = 0$. The initial position equation, on the other hand, implies

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{v}^{(1)}\alpha^{(1)} + \mathbf{v}^{(2)}\alpha^{(2)}$$

which implies $\alpha^{(1)} = 1$ and $\alpha^{(2)} = 0$. So the final solution is

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos(t)\,.$$

2. The equations of motion are given by

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) = m^2 u + u_{tt} - u_{xx} = 0\,.$$

Plugging

$$u(x,t) = e^{i(\omega t \pm kx)}$$

into this equation we get

$$(m^2 - \omega^2 + k^2)e^{i(\omega t \pm kx)} = 0$$

so the necessary relation is $\omega^2 = k^2 + m^2$.

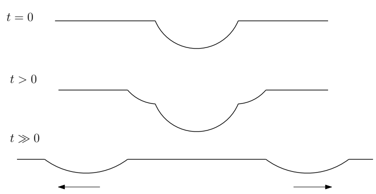
3. (a) The general solution to the wave equation is

$$u(x,t) = g(x-ct) + h(x+ct)$$

The boundary conditions tell us that $u(x,0) = g(x) + h(x) = f(x)$ and that $u_t(x,0) = -cg'(x) + ch'(x) = 0$. From the second equation we see we can choose $g(x) = h(x)$ (up to a constant shift, which will not affect the final result), in which case the first equation tells us that $g(x) = h(x) = f(x)/2$. Therefore we have that

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right)$$

The function $f(x)$ vanishes for $|x| > b$ and looks like a parabolic dip inside this region. The solution $u(x,t)$ looks like two copies of this dip each scaled by a half, one half which moves to the right with speed c and the other half moves to the left with speed c.

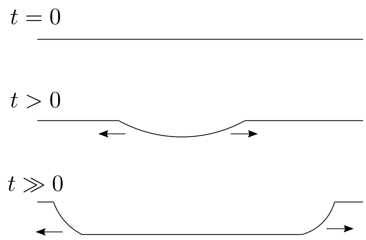


(b) In this case, substituting the general solution into the boundary conditions gives $u(x,0) = 0 = g(x) + h(x)$ and $u_t(x,0) = f(x) = -cg'(x) + ch'(x)$. We see that $g(x) = -h(x)$ and that $2ch'(x) = f(x)$, from which it follows that

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} f(s)ds.$$

This is a "spreading bump", where the height in the middle at late times is

$$\frac{1}{2c}\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2c}\int_{-b}^{b}(x^2 - b^2)dx = -\frac{2b^3}{3c}\,.$$



To see why this is the right sketch: fix a finite positive time t. Then the function

$$u(x,t) = \int_{x-ct}^{x+ct} f(s)ds$$

becomes purely a function of x, that you want to sketch. To see what the sketch is, note that $f(s)$ is only non-vanishing for s between $-b$ and b. Let me call this interval $B = [-b, b]$. On the other hand, you are integrating $f(s)$ in the interval $X(x) = [x - ct, x + ct]$ (note that this is a function of x only, as I am fixing t to a fixed value). There are three cases to consider (I will choose x positive for simplicity, it is easy to argue that $u(x,t)$ is symmetric in x because $f(s)$ is symmetric in s):

- B does not intersect $X(x)$, which happens if $x - ct > b$. In this case $u(x,t) = 0$.
- B is fully contained in $X(x)$. This happens when $x - ct < -b$ (note that because $x > 0$ this implies $ct > b$, which implies $x + ct > b$). Then

$$\frac{1}{2c}\int_{-\infty}^{\infty} f(x)dx = \frac{1}{2c}\int_{-b}^{b}(x^2 - b^2)dx = -\frac{2b^3}{3c}$$

  as above.
- For values of x between $-b + ct$ and $b + ct$ you have some function interpolating continuously (because you are integrating an continuous function on a continuously varying interval) between $-2b^3/3c$ and 0.