Fields II

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1. The equation of motion that follows from the Lagrangian, without assuming that ρ or τ are constant, follows straightforwardly from the Euler-Lagrange equations:

$$\frac{\partial}{\partial x}\left(\tau\left(\frac{\partial u}{\partial x}\right)\right)-\frac{\partial}{\partial t}\left(\rho\left(\frac{\partial u}{\partial t}\right)\right)=0$$

Substituting in a solution of the form $u(x,t) = X(x)\cos(\omega t)$ into the equation we find

$$\cos(\omega t)\frac{\partial}{\partial x}\left(\tau(x)\frac{dX}{dx}\right)=-\omega^2\rho(x)\cos(\omega t)X(x)$$

so X satisfies the ordinary differential equation

$$\frac{d}{dx}\left(\tau(x)\frac{dX}{dx}\right)+\omega^2\rho(x)X(x)=0.$$

If τ and ρ are constants this is

$$\frac{d^2X}{dx^2}+\omega^2\frac{\rho}{\tau}X=0$$

or equivalently, introducing $c^2 = \tau/\rho$

$$\frac{d^2X}{dx^2}+(\omega^2/c^2)X=0$$

which can be solved in terms of sines and cosines

$$X=\alpha\cos((\omega/c)x)+\beta\sin((\omega/c)x)$$

with α, β arbitrary constants.

2. A straightforward way of solving this problem is by plugging the given solution into $u_{tt} = u_{xx}$:

$$\sum_{n=1}^{\infty}\ddot{b}_n(t)\sin(nx)=\sum_{n=1}^{\infty}(-n^2)b_n(x)\sin(nx)\,.$$

Integrating against $\sin(mx)$ over the $[0, \pi]$ interval, and using

$$\int_0^{\pi}\sin(mx)\sin(nx)=\frac{\pi}{2}\delta_{n,m}$$

we find

$$\ddot{b}_m(t)+m^2b_m(t)=0$$

for all m. The solution to this equation is $b_n(t) = \alpha_n \cos(nt) + \beta_n \sin(nt)$ (I am changing back to an index n, this is an arbitrary naming convention), with arbitrary constants α_n, β_n (determined by initial conditions), so our general solution is

$$u(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(nt) + \beta_n \sin(nt)) \sin(nx)\,.$$

There is a second way of solving the problem which is slightly more complicated but interesting. Notice that we have a theory on an interval, with a known explicit dependence of the solution on x, so we can think of this effectively as a problem that depends on time only, with the $b_n(t)$ our generalised coordinates. We obtain the Lagrangian for the $b_n(t)$ generalised coordinates by integrating the Lagrangian density $\mathcal{L} = u_t^2 - u_x^2$ for $u(x,t)$ over the interval $x \in [0, \pi]$.

Substituting the expansion of $u(x,t)$ into the expression for $\mathcal{L}$ we find

$$\begin{aligned} L(\mathbf{b}(t), \dot{\mathbf{b}}(t)) &= \int_0^{\pi} \mathcal{L}(u, u_t, u_x) dx \\ &= \int_0^{\pi} (u_t^2 - u_x^2) dx \\ &= \int_0^{\pi} \left((\sum_{n=1}^{\infty} \dot{b}_n(t) \sin(nx))(\sum_{m=1}^{\infty} \dot{b}_m(t) \sin(mx)) \right) dx \\ &\quad - \int_0^{\pi} \left((\sum_{n=1}^{\infty} b_n(t) n \cos(nx))(\sum_{m=1}^{\infty} b_m(t) m \cos(mx)) \right) dx \end{aligned}$$

where we indicate by $\mathbf{b}(t)$ the (countably infinite) set of all $b_n(t)$. Now

$$\int_0^{\pi} \sin(mx)\sin(nx)dx = \int_0^{\pi} \cos(mx)\cos(nx) = \delta_{nm}\frac{\pi}{2}.$$

Our expression for L simplifies to

$$\begin{aligned} L &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(\dot{b}_n(t)\dot{b}_m(t)\delta_{nm}\frac{\pi}{2} \right) - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mn b_n(t) b_m(t) \delta_{nm}\frac{\pi}{2} \\ &= \frac{\pi}{2}\sum_{n=1}^{\infty} \left((\dot{b}_n(t))^2 - n^2 (b_n(t))^2 \right). \end{aligned}$$

What we see is that the string can be thought of an infinite collection of harmonic oscillators, each oscillator corresponding to a particular harmonic. What is quite nice about this example is that you can really see that the string is a system with an infinite number of degrees of freedom (in this countably infinite).

The Euler-Lagrange equations for b_n are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{b}_n}\right) - \frac{\partial L}{\partial b_n} = 0$$

This gives the equations of motion

$$\pi(\ddot{b}_n + n^2 b_n) = 0.$$

The solution to this equation is

$$b_n = \alpha_n \cos(nt) + \beta_n \sin(nt).$$

This gives us that

$$u(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos(nt) + \beta_n \sin(nt)) \sin(nx)$$

as above.

3. (a) The definition of the energy momentum tensor is

$$T_{ij} = u_i \frac{\partial \mathcal{L}}{\partial u_j} - \delta_{ij}\mathcal{L}\,.$$

Plugging in the definition of the modified Lagrangian density we get (with $x_0 \equiv t$ and $x_1 \equiv x$ as usual):

$$T_{ij} = \begin{pmatrix} \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 - m^2\cos(u) & -u_t u_x \\ u_x u_t & -\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - m^2\cos(u) \end{pmatrix}$$

(b) The equation of motion satisfied by u is given by the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) = 0$$

which in our case reads

$$-m^2 \sin(u) + u_{xx} - u_{tt} = 0$$

If we assume that $u(x,t) = f(x - ct)$ we have

$$u_{xx} = f''$$

with $f'' = \frac{d^2 f(\zeta)}{d\zeta^2}$ and

$$u_{tt} = c^2 f''$$

so that the equation satisfied by f is

$$f'' = \frac{m^2}{1-c^2}\sin(f)\,.$$

(c) For the right hand side we have

$$\frac{m^2}{1-c^2}\sin\left(4\arctan(e^{\rho\zeta})\right) = \frac{4m^2}{1-c^2}\left[\frac{e^{\rho\zeta}-e^{3\rho\zeta}}{(1+e^{2\rho\zeta})^2}\right]$$

from the relation given. For the left hand side, using the chain rule and the other relation given:

$$f'(\zeta) = 4\frac{\rho e^{\rho\zeta}}{1+e^{2\rho\zeta}}$$

and deriving once more

$$\begin{aligned} f''(\zeta) &= 4\left[\frac{\rho^2 e^{\rho\zeta}}{1+e^{2\rho\zeta}} - \frac{2\rho^2 e^{3\rho\zeta}}{(1+e^{2\rho\zeta})^2}\right] \\ &= 4\rho^2 e^{\rho\zeta}\left[\frac{1+e^{2\rho\zeta}-2e^{2\rho\zeta}}{(1+e^{2\rho\zeta})^2}\right] \\ &= 4\rho^2\left[\frac{e^{\rho\zeta}-e^{3\rho\zeta}}{(1+e^{2\rho\zeta})^2}\right] \end{aligned}$$

which implies that the equation is satisfied as long as $\rho^2 = m^2/(1-c^2)$.

(d) The soliton travels towards the right with constant speed c, keeping its shape. At $x - ct \to -\infty$ (so on the infinite left) we have

$$u(x,t) \approx 4\arctan(0) = 0$$

while on the infinite right we have

$$u(x,t) \approx 4\arctan(+\infty) = 2\pi$$

A plot of the resulting structure is shown in the figure below, it is a travelling lump of energy that brings down u from 2π to 0.

It is illuminating to compute the energy density for the soliton solution we have been discussing. A straightforward computation using for the tt component of the energy-momentum tensor computed above, and the fact that

$$\cos(4\arctan(x)) = \frac{1-6x^2+x^4}{(1+x^2)^2}$$

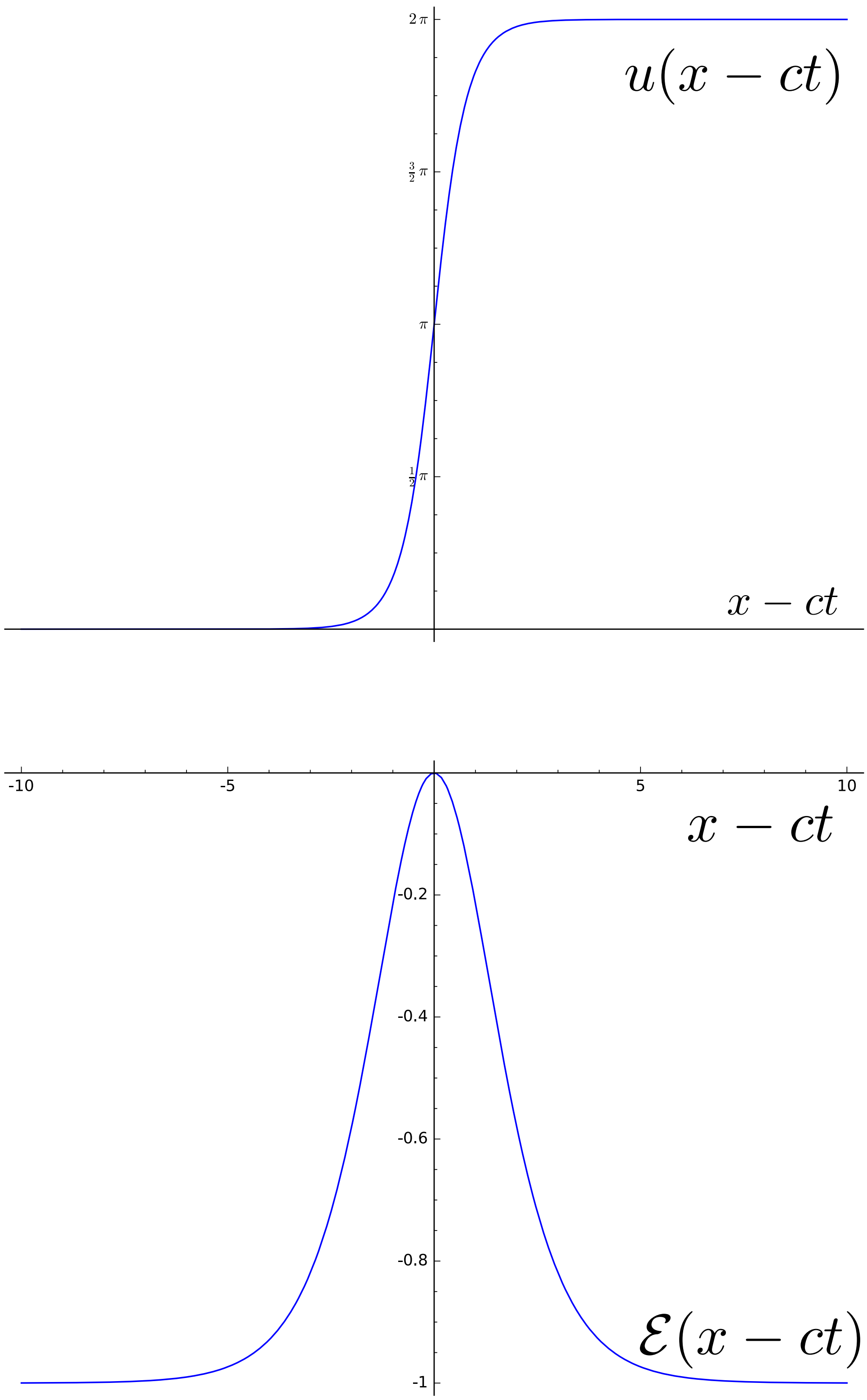
gives

$$\mathcal{E}(x,t) = T_{tt}(\zeta) = -\frac{m^2}{(1+e^{2\rho\zeta})^2}\left[1+e^{4\rho\zeta}-e^{2\rho\zeta}\left(6+8\frac{1+c^2}{1-c^2}\right)\right]$$

We have that

$$\lim_{x\to-\infty}\mathcal{E}(x,t) = \lim_{x\to\infty} = -m^2$$

which is a constant, which we might redefine away. The non-trivial part of the energy of the solitonic wave is concentrated around $x - ct = 0$, as sketched in the next figure (for some arbitrary choice of constants).


2π
$\frac{3}{2}\pi$
π
$\frac{1}{2}\pi$
$u(x-ct)$
$x-ct$
-10
-5
5
10
$x-ct$
-0.2
-0.4
-0.6
-0.8
-1
$\mathcal{E}(x-ct)$

4. We have

$$\sum_j \frac{\partial T_{ij}}{\partial x_j} = \sum_j \frac{\partial}{\partial x_j}\left(u_i \frac{\partial \mathcal{L}}{\partial u_j} - \delta_{ij}\mathcal{L}\right)$$
$$= u_i\left(\sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial u_j}\right)\right) + \sum_j \left(\frac{\partial u_i}{\partial x_j}\frac{\partial \mathcal{L}}{\partial u_j}\right) - \frac{\partial \mathcal{L}}{\partial x_i}.$$

Using the Euler-Lagrange equations on the first term this becomes

$$\sum_j \frac{\partial T_{ij}}{\partial x_j} = u_i \frac{\partial \mathcal{L}}{\partial u} + \sum_j \left(\frac{\partial u_i}{\partial x_j}\frac{\partial \mathcal{L}}{\partial u_j}\right) - \frac{\partial \mathcal{L}}{\partial x_i}.$$

and introducing $u_{ij} = \frac{\partial^2 \mathcal{L}}{\partial x_i \partial x_j} = \frac{\partial^2 \mathcal{L}}{\partial x_j \partial x_i} = u_{ji}$ this can be written as

$$\sum_j \frac{\partial T_{ij}}{\partial x_j} = u_i \frac{\partial \mathcal{L}}{\partial u} + \sum_j \left(u_{ij}\frac{\partial \mathcal{L}}{\partial u_j}\right) - \frac{\partial \mathcal{L}}{d x_i}.$$

Since we have, by the chain rule:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial \mathcal{L}}{\partial u} u_i + \sum_j \frac{\partial \mathcal{L}}{\partial u_j} u_{ij}$$

we are done:

$$\sum_j \frac{\partial T_{ij}}{\partial x_j} = 0.$$

5. (a) The definition of the T_{tt} component of the energy momentum tensor (which we identify with the energy) is

$$T_{tt} = u_t \frac{\partial \mathcal{L}}{\partial u_t} - \mathcal{L}$$

In the given Lagrangian density u_t only appears in the kinetic term:

$$\frac{\partial \mathcal{L}}{\partial u_t} = \frac{\partial \mathcal{T}}{\partial u_t} = 2u_t f(u)$$

so that

$$\begin{aligned} T_{tt} &= u_t(2u_t)f(u) - \mathcal{L} \\ &= 2\mathcal{T} - (\mathcal{T} - \mathcal{V}) \\ &= \mathcal{T} + \mathcal{V}. \end{aligned}$$

(b) Conservation of energy is

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial T_{01}}{\partial x} = 0\,.$$

From the definition of the energy-momentum tensor we have

$$T_{01} = -u_t \frac{\partial \mathcal{V}}{\partial u_x}\,.$$

So we have

$$\begin{aligned} \frac{dE_{(a,b)}}{dt} &= \frac{d}{dt}\int_a^b \mathcal{E}\,dx \\ &= \int_a^b \frac{\partial \mathcal{E}}{\partial t}\,dx \\ &= -\int_a^b \frac{\partial T_{01}}{\partial x}\,dx \\ &= -\left[T_{01}\right]_a^b \end{aligned}$$

so we have

$$\mathcal{F} = -u_t \frac{\partial \mathcal{V}}{\partial u_x}\,.$$

6. (a) The fact that $u(0,t) = 0$ implies

$$u_t(0,t) = \lim_{x\to 0^-} \frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial t}\left(\lim_{x\to 0^-} u(x,t)\right) = \frac{\partial}{\partial t}(0) = 0$$

(assuming that u is regular enough). The energy flux entering the boundary from the left for a one-dimensional string is given by

$$T_{tx}(0,t) = -\tau u_x(0,t)u_t(0,t) = -\tau u_x(0,t)\cdot 0 = 0\,.$$

Now consider the given ansatz

$$u(x,t) = \Re\left((e^{ikx} + Re^{-ikx})e^{-ikct}\right)$$

The boundary condition $u(0,t) = 0$ implies

$$\Re\left((1+R)e^{-ikct}\right) = 0$$

which is satisfied for all t if and only if $R = -1$.

(b) The boundary condition reads, in terms of f and g:

$$f(-ct) + g(ct) = 0\,.$$

This should holds for all t, so we find that $g(\zeta) = -f(-\zeta)$, for any ζ. D'Alembert's solution then becomes

$$u(x,t) = f(x-ct) - f(-x-ct)\,.$$

For any fixed t this is an odd function.

(c) The case of Neumann boundary conditions is $u_x(0,t) = 0$. The energy flux into the boundary then clearly vanishes:

$$T_{tx}(0,t) = -\tau u_x(0,t)u_t(0,t) = -\tau \cdot 0 \cdot u_t(0,t) = 0\,.$$

For the case of the given ansatz

$$u(x,t) = \Re\left((e^{ikx} + Re^{-ikx})e^{-ikct}\right)$$

we have

$$u_x(x,t) = \Re\left((ike^{ikx} - Rike^{-ikx})e^{-ikct}\right)$$

so that

$$u_x(0,t) = \Re\left(ik(1-R)e^{-ikct}\right)$$

which is satisfied for all t if and only if $R = 1$.

(d) We have

$$f'(-ct) + g'(ct) = 0$$

which implies that for any ζ we have $f'(\zeta) = -g'(-\zeta)$. By integration, this implies $f(\zeta) = g(-\zeta) + p$, for some constant p. We thus find the general solution in this case to be

$$u(x,t) = g(x+ct) + g(ct-x) + p\,.$$

(We could absorb p into the definition of g, if we like.) At any fixed t this is a symmetric function around $x = 0$.

7. All the results follow from applying the chain rule, and using the fact that $F_i(u,x,t)$ depends on u, x, t but not on u_x or u_t. For instance

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}u_x$$

so from here

$$\frac{\partial\left(\frac{dF}{dx}\right)}{\partial u_x} = \frac{\partial}{\partial u_x}\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}u_x\right) = \frac{\partial F}{\partial u}$$

using the fact that neither $\frac{\partial F}{\partial x}$ nor $\frac{\partial F}{\partial u}$ depend on u_x. The other identities of this kind can be proved similarly. For the second class of Lemmas we use the chain rule and that partial derivatives commute:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\partial F}{\partial u}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}\right)u_x \\
&= \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u}\right) + \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}u_x\right) \\
&= \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial x}\right) + \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}u_x\right) \\
&= \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}u_x\right) \\
&= \frac{\partial}{\partial u}\left(\frac{dF}{dx}\right)
\end{aligned}$$

With these lemmas in hand it is easy to prove what we want:

$$\frac{\partial\left(\frac{dF_2}{dt}\right)}{\partial u} - \frac{d}{dt}\left(\frac{\partial\left(\frac{dF_2}{dt}\right)}{\partial u_t}\right) - \frac{d}{dx}\left(\frac{\partial\left(\frac{dF_2}{dt}\right)}{\partial u_x}\right) = \frac{d}{dt}\left(\frac{\partial F_2}{\partial u}\right) - \frac{d}{dt}\left(\frac{\partial F_2}{\partial u}\right) - \frac{d}{dx}(0) = 0$$

and similarly for $\frac{dF_1(u,x,t)}{dx}$.