Week 9 problems

1. Differentiating X with respect to t we find
$$\begin{aligned}\frac{dX(a,b)}{dt} &= \int_a^b \frac{\partial}{\partial t}\left((u_t)^3 + 3u_t(u_x)^2\right)dx \\ &= \int_a^b 3(u_t)^2u_{tt} + 3u_{tt}(u_x)^2 + 6u_tu_xu_{tx}dx \\ &= \int_a^b 3(u_t)^2u_{xx} + 3u_{xx}(u_x)^2 + 6u_tu_xu_{tx}dx \\ &= \int_a^b \frac{\partial}{\partial x}\left(3(u_t)^2u_x + (u_x)^3\right)dx \\ &= \left[3(u_t)^2u_x + (u_x)^3\right]_a^b.\end{aligned}$$
This is in the desired form with $g(u, u_t, u_x) = 3(u_t)^2u_x + (u_x)^3$. Provided that $u_x, u_t \to 0$ as $|x| \to \infty$ we can show that
$$\frac{dX(-\infty,\infty)}{dt} = \left[3(u_t)^2u_x + (u_x)^3\right]_{-\infty}^{\infty} = 0.$$
so that $X(-\infty,\infty)$ is conserved.

2. (a) Calculating the time derivative of the energy we find that
$$\frac{d}{dt}E(a,b) = \int_a^b \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx$$
Now we can use the wave equation write this as
$$\begin{aligned}\frac{d}{dt}E(a,b) &= \int_a^b \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx \\ &= \int_a^b \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)dx \\ &= \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_a^b.\end{aligned}$$
(b) If $u(x,t) = f(x+t) - f(x-t)$, then $u_t = f'(x+t) + f'(x-t)$ and $u_x = f'(x+t) - f'(x-t)$. It follows that the energy
$$\begin{aligned}E(a,b) &= \frac{1}{2}\int_a^b (u_t)^2 + (u_x)^2dx \\ &= \frac{1}{2}\int_a^b (f'(x+t) + f'(x-t))^2 + (f'(x+t) - f'(x-t))^2dx \\ &= \int_a^b (f'(x+t))^2 + (f'(x-t))^2dx.\end{aligned}$$

(c) In the given case we have that $f(s) = \exp(-s^2/2)$, so that $f'(s) = -s\exp(-s^2/2)$. Putting this into the formula for E we find

$$E(0,\infty) = \int_0^\infty (x+t)^2 e^{-(x+t)^2} + (x-t)^2 e^{-(x-t)^2} dx.$$

If we change the variables of integration in the first term to $x' = -x$ we see that

$$\begin{aligned} E(0,\infty) &= \int_{-\infty}^0 (-x'+t)^2 e^{-(-x'+t)^2} dx' + \int_0^\infty (x-t)^2 e^{-(x-t)^2} dx \\ &= \int_{-\infty}^\infty (x-t)^2 e^{-(x-t)^2} dx. \end{aligned}$$

Finally changing variables to $p = x - t$ we see that

$$E(0,\infty) = \int_{-\infty}^\infty p^2 e^{-p^2} dp$$

Notice that we have already shown that $E(0,\infty)$ is independent of time! We can evaluate this integral using integration by parts

$$\int_{-\infty}^\infty p^2 e^{-p^2} dp = \int_{-\infty}^\infty p(pe^{-p^2})dp = \left[-\frac{p}{2}e^{-p^2}\right]_{-\infty}^\infty + \frac{1}{2}\int_{-\infty}^\infty e^{-p^2} dp = \frac{\sqrt{\pi}}{2}.$$

(d) The equation in part (a) tells us that

$$\frac{d}{dt}E(0,\infty) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^\infty.$$

or in terms of f

$$\begin{aligned} \frac{d}{dt}E(0,\infty) &= [(f'(x+t) + f'(x-t))(f'(x+t) - f'(x-t))]_0^\infty \\ &= \left[(f'(x+t))^2 - (f'(x-t))^2\right]_0^\infty \\ &= f'(-t)^2 - f'(t)^2 = 0. \end{aligned}$$

3. By conservation of energy, the energy stored in the boundary $E_b(t)$ changes only due to the incoming flux from the left:

$$\frac{dE_b(t)}{dt} = \lim_{x\to 0^-} T_{tx}(x,t)$$

The energy stored at the boundary is given by the extension of the spring:

$$E_b(t) = \frac{1}{2}\kappa u(0,t)^2$$

so that

$$\frac{dE_b(t)}{dt} = \kappa u(0,t) u_t(0,t)\,.$$

On the other hand, for the one-dimensional string:

$$T_{tx}(x,t) = -\tau u_x(x,t) u_t(x,t)$$

so that

$$\lim_{x\to 0^-} T_{tx}(x,t) = -\tau u_x(0,t) u_t(0,t)$$

so the energy conservation equation simplifies to

$$\kappa u(0,t) = -\tau u_x(0,t)\,.$$

For the case of the given ansatz

$$u(x,t) = \Re\left((e^{ipx} + Re^{-ipx})e^{-ipct}\right)$$

conservation of energy becomes

$$\kappa \Re\left((1+R)e^{-ipct}\right) = -\tau \Re\left(ip(1-R)e^{-ipct}\right)$$

which holds for all t if and only if

$$\kappa(1+R) = -\tau ip(1-R)$$

and a little bit of algebra then leads to

$$R = \frac{ip + \kappa/\tau}{ip - \kappa/\tau}\,.$$

As a couple of simple checks, note that for $\kappa \to \infty$, we should expect to have effective Dirichlet boundary conditions (since it costs infinite energy to extend the spring). And indeed, in this case $R = -1$. Similarly, for $\kappa \to 0$ the spring is effectively not there, so we expect to reproduce the result from having Neumann boundary conditions, $R = +1$.