Week 9 problems

1. Differentiating X with respect to t we find

$$\frac{dX(a,b)}{dt} = \int_{a}^{b} \frac{\partial}{\partial t} \left((u_{t})^{3} + 3u_{t}(u_{x})^{2} \right) dx$$

$$= \int_{a}^{b} 3(u_{t})^{2} u_{tt} + 3u_{tt}(u_{x})^{2} + 6u_{t}u_{x}u_{tx}dx$$

$$= \int_{a}^{b} 3(u_{t})^{2} u_{xx} + 3u_{xx}(u_{x})^{2} + 6u_{t}u_{x}u_{tx}dx$$

$$= \int_{a}^{b} \frac{\partial}{\partial x} \left(3(u_{t})^{2} u_{x} + (u_{x})^{3} \right) dx$$

$$= \left[3(u_{t})^{2} u_{x} + (u_{x})^{3} \right]_{a}^{b}.$$

This is in the desired form with $g(u, u_t, u_x) = 3(u_t)^2 u_x + (u_x)^3$. Provided that $u_x, u_t \to 0$ as $|x| \to \infty$ we can show that

$$\frac{dX(-\infty,\infty)}{dt} = \left[3(u_t)^2 u_x + (u_x)^3\right]_{-\infty}^{\infty} = 0$$

so that $X(-\infty,\infty)$ is conserved.

2. 2.1. Calculating the time derivative of the energy we find that

$$\frac{d}{dt}E(a,b) = \int_{a}^{b} \frac{\partial u}{\partial t} \frac{\partial^{2} u}{\partial t^{2}} + \frac{\partial u}{\partial x} \frac{\partial^{2} u}{\partial x \partial t} dx$$

Now we can use the wave equation write this as

$$\frac{d}{dt}E(a,b) = \int_{a}^{b} \frac{\partial u}{\partial t} \frac{\partial^{2} u}{\partial x^{2}} + \frac{\partial u}{\partial x} \frac{\partial^{2} u}{\partial x \partial t} dx$$
$$= \int_{a}^{b} \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \frac{\partial u}{\partial x}\right) dx$$
$$= \left[\frac{\partial u}{\partial t} \frac{\partial u}{\partial x}\right]_{a}^{b}.$$

2.2. If u(x,t) = f(x+t) - f(x-t), then $u_t = f'(x+t) + f'(x-t)$ and $u_x = f'(x+t) - f'(x-t)$. It follows that the energy

$$E(a,b) = \frac{1}{2} \int_{a}^{b} (u_{t})^{2} + (u_{x})^{2} dx$$

= $\frac{1}{2} \int_{a}^{b} (f'(x+t) + f'(x-t))^{2} + (f'(x+t) - f'(x-t))^{2} dx$
= $\int_{a}^{b} (f'(x+t))^{2} + (f'(x-t))^{2} dx.$

- MATH2071 Mathematical Physics II Solutions Week 9
- 2.3. In the given case we have that $f(s) = \exp(-s^2/2)$, so that $f'(s) = -s \exp(-s^2/2)$. Putting this into the formula for E we find

$$E(0,\infty) = \int_0^\infty (x+t)^2 e^{-(x+t)^2} + (x-t)^2 e^{-(x-t)^2} dx.$$

If we change the variables of integration in the first term to x' = -x we see that

$$E(0,\infty) = \int_{-\infty}^{0} (-x'+t)^2 e^{-(-x'+t)^2} dx' + \int_{0}^{\infty} (x-t)^2 e^{-(x-t)^2} dx$$
$$= \int_{-\infty}^{\infty} (x-t)^2 e^{-(x-t)^2} dx.$$

Finally changing variables to p = x - t we see that

$$E(0,\infty) = \int_{-\infty}^{\infty} p^2 e^{-p^2} dp$$

Notice that we have already shown that $E(0, \infty)$ is independent of time! We can evaluate this integral using integration by parts

$$\int_{-\infty}^{\infty} p^2 e^{-p^2} dp = \int_{-\infty}^{\infty} p(p e^{-p^2}) dp = \left[-\frac{p}{2} e^{-p^2} \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-p^2} dp = \frac{\sqrt{\pi}}{2}.$$

2.4. The equation in part (a) tells us that

$$\frac{d}{dt}E(0,\infty) = \left[\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right]_0^\infty.$$

or in terms of f

$$\frac{d}{dt}E(0,\infty) = \left[(f'(x+t) + f'(x-t))(f'(x+t) - f'(x-t)) \right]_0^\infty$$

= $\left[(f'(x+t))^2 - (f'(x-t))^2 \right]_0^\infty$
= $f'(-t)^2 - f'(t)^2 = 0.$

3. By conservation of energy, the energy stored in the boundary $E_b(t)$ changes only due to the incoming flux from the left:

$$\frac{dE_b(t)}{dt} = \lim_{x \to 0^-} T_{tx}(x, t)$$

The energy stored at the boundary is given by the extension of the spring:

$$E_b(t) = \frac{1}{2}\kappa u(0,t)^2$$

so that

$$\frac{dE_b(t)}{dt} = \kappa u(0,t)u_t(0,t) \,.$$

On the other hand, for the one-dimensional string:

$$T_{tx}(x,t) = -\tau u_x(x,t)u_t(x,t)$$

so that

$$\lim_{x \to 0^{-}} T_{tx}(x,t) = -\tau u_x(0,t) u_t(0,t)$$

so the energy conservation equation simplifies to

$$\kappa u(0,t) = -\tau u_x(0,t) \,.$$

For the case of the given ansatz

$$u(x,t) = \Re \left((e^{ipx} + Re^{-ipx})e^{-ipct} \right)$$

conservation of energy becomes

$$\kappa \Re \left((1+R)e^{-ipct} \right) = -\tau \Re \left(ip(1-R)e^{-ipct} \right)$$

which holds for all t if and only if

$$\kappa(1+R) = -\tau i p(1-R)$$

and a little bit of algebra then leads to

$$R = \frac{ip + \kappa/\tau}{ip - \kappa/\tau} \,.$$

As a couple of simple checks, note that for $\kappa \to \infty$, we should expect to have effective Dirichlet boundary conditions (since it costs infinite energy to extend the spring). And indeed, in this case R = -1. Similarly, for $\kappa \to 0$ the spring is effectively not there, so we expect to reproduce the result from having Neumann boundary conditions, R = +1.