Quantum Computing Epiphany Assignment 1 Solutions

Note that the solutions to all parts below are certainly not unique.

1. There are four possible values for x, and $f(x)$ has two possible values for each choice, so there are $2^4 = 16$ functions which we can label $f_0, f_1, \ldots, f_{15}$. I have defined these function in the table below. Note that the list of all outputs for f_N is just N written as a 4-digit binary in these conventions – this is a simple systematic way to include all possible functions and to ensure you don't duplicate any functions.

   Note that the table listing the output values is by far the simplest way to answer the question. If you use a description in terms of logical operations, since these expressions are not unique, you cannot immediately tell if you've defined the same function twice (and hence missed another function if you are relying on counting the number of functions).

2. I have not drawn the circuits but instead defined them in the text in the table below. To answer the question correctly, you should draw the circuits (or give a detailed explanation of how they are constructed as I do now). Again, these are not unique so if you have another circuit which does the same, that is fine, although it may be useful to spot easy ways to simplify circuits. E.g. I have seen examples with two $C^m NOT$ gates with the same control(s) and target following each other – that is just the identity so both can be removed. Another example is including an extra ancillary bit and simply copying (with a CNOT) what could be the output bit to that bit – a similar example is unnecessarily making copies of the input bits.

   For the circuits we can take 3 bits in total, the two input bits and another bit initialised to 0 which will give the output bit – it is not necessary to include any further (ancillary) bits. (Some circuits can be drawn using just the two input bits and clearly identifying which gives the output – this is fine, but conventionally we will keep the input and output lines separate.) Taking $x = (x_1 x_0)_2$, we can write $CCNOT$ to mean a CCNOT gate with the output bit as the target and the two input bits (x_1 and x_0) as the controls, $CNOT_0$ ($CNOT_1$) to mean $CNOT$ acting on the output bit controlled by x_0 (x_1), and NOT to mean a NOT acting on the output. (If you give some statements without this level of detail, it is not clear what your circuits are – anyway it is probably simplest just to draw the circuits.) You can then easily draw the circuits by placing these gates in the same order left to right. (Actually, if you use these gates only, the order does not matter

– in general the order is important!) Note that you were only asked to draw the 8 functions which satisfy $f(00) = 0$. In this notation that is the functions $f_0, f_1, \ldots, f_7$.

3. Note that the functions with $f(00) = 1$ (in this notation $f_8, f_9, \ldots, f_{15}$) are the *NOT* of a function from the first half, specifically f_{15-N} is related to f_N in this way. So, if you have constructed circuits for the functions with $f(00) = 0$, you can simply include a NOT gate at the end of the output to produce the remaining circuits. If you have used the circuits described above, the NOT gate can be placed anywhere on the output line – but note this is not true in general so the placement on the output line might matter if you drew different circuits.

   To answer the question fully it is not sufficient to simply state that you take NOT of the function – you need to indicate (at least with an example, but better to be general) where to put the NOT gate.

Here is the table, also including one expression for the functions in terms of logical operations.

x	00	01	10	11	Representation	Logic output
f_0	0	0	0	0	$Trivial$	0
f_1	0	0	0	1	$CCNOT$	$x_0\ AND\ x_1$
f_2	0	0	1	0	$CCNOT\ CNOT_1$	$(NOT\ x_0)\ AND\ x_1$
f_3	0	0	1	1	$CNOT_1$	x_1
f_4	0	1	0	0	$CCNOT\ CNOT_0$	$x_0\ AND\ (NOT\ x_1)$
f_5	0	1	0	1	$CNOT_0$	x_0
f_6	0	1	1	0	$CNOT_0\ CNOT_1$	$x_0\ XOR\ x_1$
f_7	0	1	1	1	$CCNOT\ CNOT_0\ CNOT_1$	$x_0\ OR\ x_1$
f_8	1	0	0	0	$CCNOT\ CNOT_0\ CNOT_1\ NOT$	$x_0\ NOR\ x_1$
f_9	1	0	0	1	$CNOT_0\ CNOT_1\ NOT$	$x_0\ NXOR\ x_1$
f_{10}	1	0	1	0	$CNOT_0\ NOT$	$NOT\ x_0$
f_{11}	1	0	1	1	$CCNOT\ CNOT_0\ NOT$	$x_0\ NAND\ (NOT\ x_1)$
f_{12}	1	1	0	0	$CNOT_1\ NOT$	$NOT\ x_1$
f_{13}	1	1	0	1	$CCNOT\ CNOT_1\ NOT$	$(NOT\ x_0)\ NAND\ x_1$
f_{14}	1	1	1	0	$CCNOT\ NOT$	$x_0\ NAND\ x_1$
f_{15}	1	1	1	1	NOT	1