Quantum Computing Epiphany Assignment 2 Solutions

1. This is very straightforward to calculate directly for each computational basis state.

$$\begin{aligned} |00\rangle &\to |10\rangle \to |11\rangle \to & |01\rangle \\ |01\rangle &\to |11\rangle \to |10\rangle \to & |00\rangle \\ |10\rangle &\to |00\rangle \to |00\rangle \to & |10\rangle \\ |11\rangle &\to |01\rangle \to |01\rangle \to & |11\rangle \end{aligned}$$

   Alternatively, note that two NOT gates act on q_1 so it is unchanged. (If we input computational basis states, the state remains a computational basis state throughout the circuit so we can be sure the CNOT gate does not act on the control qubit.) As it is used as the control after the first NOT, q_0 is changed precisely when initially $q_1 = 0$.

   Although this is probably the most intuitive way to present the circuit, there is a slightly simpler alternative using only two gates. Instead of the two X gates acting on qubit 1, place a single X gate acting on qubit 0 (before or after the CNOT gate). This works because $X^2 = I$ so when $q_1 = 1$ nothing happens to qubit 0 but when $q_1 = 0$ X acts on qubit 0.

2. Again, note that if we input computational basis states, the state remains in a computational basis state throughout the circuit.

   If q_2 is zero, the circuit simplifies to just two CNOTs (control q_1, target q_0), which is trivial.

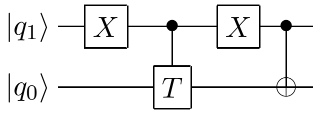
   For $q_2 = 1$ you can directly calculate

$$\begin{aligned} |10q_0\rangle \to &\quad |10q_0\rangle \to |11q_0\rangle \to |11(q_0 \oplus 1)\rangle \to |10(q_0 \oplus 1)\rangle \to & |10q_0\rangle \\ |11q_0\rangle \to &\quad |11(q_0 \oplus 1)\rangle \to |10(q_0 \oplus 1)\rangle \to |10(q_0 \oplus 1)\rangle \to |11(q_0 \oplus 1)\rangle \to & |11q_0\rangle\ . \end{aligned}$$

   Alternatively, for $q_2 = 1$ note that q_2 and q_1 are not changed, since for q_1 we have two NOTs which gives the identity. For q_0 since q_1 has a NOT between the two $CNOT$s where q_1 is the control, exactly one of them will act as NOT on q_0. However, the final $CNOT$ with control $q_2 = 1$ acts as another NOT on q_0, so it is also unchanged.

   Thus, the action in the computational basis is completely trivial. This is a trivial unitary. The circuit can then be simplified to simply 3 horizontal lines.

3. Acting on 2-qubit computational basis states $|q_1 q_0\rangle$, this is T on $|q_0\rangle$ if $q_1 = 0$, and X on $|q_0\rangle$ if $q_1 = 1$. Hence we want



It is also correct to have the CNOT gate on the left. Also, like in question 1, it is possible to remove the two X gates acting on qubit 1, but now replacing the controlled T gate with a controlled $T^\dagger$ gate and including a T gate acting on qubit 0.

Note that in this question you don't really need to use the general procedure outlined in the lectures to decompose the 4×4 unitary matrix since it obviously decomposes as $U = \begin{pmatrix} I & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}$ and there is no need to use a Gray code to permute the basis states.