Complex hyperbolic quasi-Fuchsian groups

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For Bill Harvey on the occasion of his 65th birthday

Abstract

A complex hyperbolic quasi-Fuchsian group is a discrete, faithful, type preserving and geometrically finite representation of a surface group as a subgroup of the group of holomorphic isometries of complex hyperbolic space. Such groups are direct complex hyperbolic generalisations of quasi-Fuchsian groups in three dimensional (real) hyperbolic geometry. In this article we present the current state of the art of the theory of complex hyperbolic quasi-Fuchsian groups.

1 Introduction

The purpose of this paper is to outline what is known about the complex hyperbolic analogue of quasi-Fuchsian groups. Discrete groups of complex hyperbolic isometries have not been studied as widely as their real hyperbolic

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counterparts. Nevertheless, they are interesting to study and should be more widely known. The classical theory of quasi-Fuchsian groups serves a model for the complex hyperbolic theory, but results do not usually generalise in a straightforward way. This is part of the interest of the subject.

Complex hyperbolic Kleinian groups were first studied by Picard at about the same time as Poincaré was developing the theory of Fuchsian and Kleinian groups. In spite of work by several other people, including Giraud and Cartan, the complex hyperbolic theory did not develop as rapidly as the real hyperbolic theory. So, by the time Ahlfors and Bers were laying the foundations for the theory of quasi-Fuchsian groups, complex hyperbolic geometry was hardly studied at all. Later, work of Chen and Greenberg and of Mostow on symmetric spaces led to a resurgence of interest in complex hyperbolic discrete groups. The basic theory of complex hyperbolic quasi-Fuchsian groups was laid out by Goldman and these foundations have been built upon by many other people.

There are several sources of material on complex hyperbolic geometry. The book of Goldman [27] gives an encyclopedic source of many facts, results and proofs about complex hyperbolic geometry. The forthcoming book of Parker [47] is intended to give a gentler background to the subject, focusing on discrete groups. The book of Schwartz [61] also gives a general introduction, but concentrates on the proof and application of a particular theorem. Additionally, most of the papers in the bibliography contain some elementary material but they often use different conventions and notation. Therefore we have tried to make this paper as self contained as possible, and we hope that it will become a useful resource for readers who want to begin studying complex hyperbolic quasi-Fuchsian groups.

This paper is organised as follows. We give a wide ranging introduction to complex hyperbolic geometry in Section 2. We then go on to discuss the geometry of complex hyperbolic surface group representations in Section 3, including the construction of fundamental domains. One of the most striking aspects of this theory is that, unlike the real hyperbolic case, there is a radical difference between the case where our surface has punctures or is closed (and without boundary). We discuss these two cases separately in Sections 4 and 5. Finally, in Section 6 we give some open problems and conjectures.

It is a great pleasure for us to present this survey as a contribution to a volume in honour of Bill Harvey. Bill's contributions to the classical theory of Teichmüller and quasi-Fuchsian spaces have been an inspiration to us and, more importantly, he is a good friend to both of us.

2 Complex Hyperbolic Space

2.1 Models of complex hyperbolic space

The material in this section is standard. Further details may be found in the books [27] or [47]. Let $\mathbb{C}^{2,1}$ be the complex vector space of complex dimension 3 equipped with a non-degenerate, indefinite Hermitian form $\langle\cdot,\cdot\rangle$ of signature $(2,1)$, that is $\langle\cdot,\cdot\rangle$ is given by a non-singular 3×3 Hermitian matrix H with 2 positive eigenvalues and 1 negative eigenvalue. There are two standard matrices H which give different Hermitian forms on $\mathbb{C}^{2,1}$. Following Epstein, see [14], we call these the first and second Hermitian forms. Let $\mathbf{z}$ and $\mathbf{w}$ be the column vectors $[z_1,\, z_2,\, z_3]^t$ and $[w_1,\, w_2,\, w_3]^t$ respectively. The *first Hermitian form* is defined to be:

$$\langle \mathbf{z},\mathbf{w}\rangle_1 = \mathbf{w}^* H_1 \mathbf{z} = z_1\overline{w}_1 + z_2\overline{w}_2 - z_3\overline{w}_3$$

where H_1 is the Hermitian matrix:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The *second Hermitian form* is defined to be:

$$\langle \mathbf{z},\mathbf{w}\rangle_2 = \mathbf{w}^* H_2 \mathbf{z} = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1$$

where H_2 is the Hermitian matrix:

$$H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

There are other Hermitian forms which are widely used in the literature.

Given any two Hermitian forms of signature $(2,1)$ we can pass between them using a *Cayley transform*. This is not unique for we may precompose and postcompose by any unitary matrix preserving the relevant Hermitian form. For example, one may check directly that the following Cayley transform interchanges the first and second Hermitian forms:

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

In what follows we shall use subscripts only if it is necessary to specify which Hermitian form to use. When there is no subscript, then the reader can use either of these (or her/his favourite Hermitian form on $\mathbb{C}^3$ of signature $(2,1)$).

Since $\langle \mathbf{z}, \mathbf{z} \rangle$ is real for all $\mathbf{z} \in \mathbb{C}^{2,1}$ we may define subsets V_-, V_0 and V_+ of $\mathbb{C}^{2,1}$ by

$$\begin{aligned} V_- &= \left\{ \mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle < 0 \right\}, \\ V_0 &= \left\{ \mathbf{z} \in \mathbb{C}^{2,1} - \{0\} | \langle \mathbf{z}, \mathbf{z} \rangle = 0 \right\}, \\ V_+ &= \left\{ \mathbf{z} \in \mathbb{C}^{2,1} | \langle \mathbf{z}, \mathbf{z} \rangle > 0 \right\}. \end{aligned}$$

We say that $\mathbf{z} \in \mathbb{C}^{2,1}$ is *negative, null* or *positive* if $\mathbf{z}$ is in V_-, V_0 or V_+ respectively. Motivated by special relativity, these are sometimes called time-like, light-like and space-like. Let $\mathbb{P} : \mathbb{C}^{2,1} \longmapsto \mathbb{CP}^2$ be the standard projection map. On the chart of $\mathbb{C}^{2,1}$ where $z_3 \neq 0$ this projection map is given by

$$\mathbb{P} : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{pmatrix} z_1/z_3 & z_2/z_3 \end{pmatrix} \in \mathbb{C}^2.$$

Because $\langle \lambda\mathbf{z}, \lambda\mathbf{z} \rangle = |\lambda|^2 \langle \mathbf{z}, \mathbf{z} \rangle$ we see that for any non-zero complex scalar λ the point $\lambda\mathbf{z}$ is negative, null or positive if and only if $\mathbf{z}$ is. It makes sense to describe $\mathbb{P}\mathbf{z} \in \mathbb{CP}^2$ as positive, null or negative.

Definition 2.1 *The projective model of complex hyperbolic space is defined to be the collection of negative lines in $\mathbb{C}^{2,1}$ and its boundary is defined to be the collection of null lines. In other words $\mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_-$ and $\partial\mathbf{H}^2_{\mathbb{C}}$ is $\mathbb{P}V_0$.*

We define the other two standard models of complex hyperbolic space by taking the section of $\mathbb{C}^{2,1}$ defined by $z_3 = 1$ and considering $\mathbb{P}V_-$ for the first and second Hermitian forms. In particular, we define the *standard lift* of $z = (z_1, z_2) \in \mathbb{C}^2$ to be $\mathbf{z} = [z_1, z_2, 1]^t \in \mathbb{C}^{2,1}$. It is clear that $\mathbb{P}\mathbf{z} = z$. Points in complex hyperbolic space will be those $z \in \mathbb{C}^2$ for which their standard lift satisfies $\langle \mathbf{z}, \mathbf{z} \rangle < 0$. Taking the first and second Hermitian forms, this gives two models of complex hyperbolic space which naturally generalise, respectively, the Poincaré disc and half plane models of the hyperbolic plane.

For the first Hermitian form we obtain $z = (z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}}$ provided

$$\langle \mathbf{z}, \mathbf{z} \rangle_1 = z_1\overline{z}_1 + z_2\overline{z}_2 - 1 < 0$$

or, in other words $|z_1|^2 + |z_2|^2 < 1$.

Definition 2.2 *The unit ball model of complex hyperbolic space* $\mathbf{H}^2_{\mathbb{C}}$ *is*

$$\mathbb{B}^2 = \Big\{ z = (z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1|^2 + |z_2|^2 < 1 \Big\}.$$

Its boundary $\partial \mathbf{H}^2_{\mathbb{C}}$ *is the unit sphere*

$$S^3 = \Big\{ z = (z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 1 \Big\}.$$

For the second Hermitian form we obtain $z = (z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}}$ provided

$$\langle \mathbf{z}, \mathbf{z} \rangle_2 = z_1 + z_2 \overline{z}_2 + \overline{z}_1 < 0.$$

In other words $2\Re(z_1) + |z_2|^2 < 0$.

Definition 2.3 *The Siegel domain model of complex hyperbolic space* $\mathbf{H}^2_{\mathbb{C}}$ *is*

$$\mathbb{S}^2 = \Big\{ z = (z_1, z_2) \in \mathbb{C}^2 \ : \ 2\Re(z_1) + |z_2|^2 < 0 \Big\}.$$

Its boundary is the one point compactification of $\mathbb{R}^3$. *It turns out that this is naturally endowed with the group structure of the Heisenberg group* $\mathfrak{N}$. *That is* $\partial \mathbf{H}^2_{\mathbb{C}} = \mathfrak{N} \cup \{\infty\}$ *where*

$$\mathfrak{N} = \Big\{ z = (z_1, z_2) \in \mathbb{C}^2 \ : \ 2\Re(z_1) + |z_2|^2 = 0 \Big\}.$$

The standard lift of ∞ *is the column vector* $[1,\ 0,\ 0]^t \in \mathbb{C}^{2,1}$.

2.2 Bergman metric

The *Bergman metric* on $\mathbf{H}^2_{\mathbb{C}}$ is defined by

$$ds^2 = -\frac{4}{\langle \mathbf{z}, \mathbf{z} \rangle^2} \det \begin{bmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle d\mathbf{z}, \mathbf{z} \rangle \\ \langle \mathbf{z}, d\mathbf{z} \rangle & \langle d\mathbf{z}, d\mathbf{z} \rangle \end{bmatrix}.$$

Alternatively, it is given by the distance function ρ given by the formula

$$\cosh^2 \left(\frac{\rho(z, w)}{2} \right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle} = \frac{|\langle \mathbf{z}, \mathbf{w} \rangle|^2}{|\mathbf{z}|^2 |\mathbf{w}|^2}$$

where $\mathbf{z}$ and $\mathbf{w}$ in V_- are the standard lifts of z and w in $\mathbf{H}^2_{\mathbb{C}}$ and $|\mathbf{z}| = \sqrt{-\langle \mathbf{z}, \mathbf{z} \rangle}$.

Substituting the first Hermitian form in these formulae gives the following expressions for the Bergman metric and distance function for the unit ball model

$$\begin{aligned} ds^2 &= \frac{-4}{(|z_1|^2+|z_2|^2-1)^2} \det \begin{pmatrix} |z_1|^2+|z_2|^2-1 & \overline{z}_1 d\,z_1 + \overline{z}_2 d\,z_2 \\ z_1 d\,\overline{z}_1 + z_2 d\,\overline{z}_2 & |d\,z_1|^2+|d\,z_2|^2 \end{pmatrix} \\ &= \frac{4(|dz_1|^2+|dz_2|^2-|z_1dz_2-z_2dz_1|^2)}{(1-|z_1|^2-|z_2|^2)^2} \end{aligned}$$

and

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{|1-z_1\overline{w}_1 - z_2\overline{w}_2|^2}{(1-|z_1|^2-|z_2|^2)(1-|w_1|^2-|w_2|^2)}.$$

Likewise, the Bergman metric on the Siegel domain is given by

$$\begin{aligned} ds^2 &= \frac{-4}{(z_1+|z_2|^2+\overline{z}_1)^2} \det \begin{pmatrix} z_1+|z_2|^2+\overline{z}_1 & d\,z_1 + \overline{z}_2 d\,z_2 \\ d\,\overline{z}_1 + z_2 d\,\overline{z}_2 & |d\,z_2|^2 \end{pmatrix} \\ &= \frac{-4(z_1+|z_2|^2+\overline{z}_1)|d\,z_2|^2 + 4|d\,z_1+\overline{z}_2 d\,z_2|^2}{(z_1+|z_2|^2+\overline{z}_1)^2}. \end{aligned}$$

The corresponding distance formula is

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{|z_1+\overline{w}_1+z_2\overline{w}_2|^2}{(z_1+\overline{z}_1+|z_2|^2)(w_1+\overline{w}_1+|w_2|^2)}.$$

2.3 Isometries

Let $\mathrm{U}(2,1)$ be the group of matrices that are unitary with respect to the form $\langle\cdot,\cdot\rangle$ corresponding to the Hermitian matrix H. By definition, each such matrix A satisfies the relation $A^*HA = H$ and hence $A^{-1} = H^{-1}A^*H$, where $A^* = \overline{A}^T$.

The group of holomorphic isometries of complex hyperbolic space is the *projective unitary group* $\mathrm{PU}(2,1) = \mathrm{U}(2,1)/\mathrm{U}(1)$, where we make the natural identification $\mathrm{U}(1) = \{e^{i\theta}I, \theta \in [0,2\pi)\}$ and I is the 3×3 identity matrix. The full group of complex hyperbolic isometries is generated by $\mathrm{PU}(2,1)$ and the conjugation map $(z_1,z_2) \longmapsto (\overline{z}_1,\overline{z}_2)$.

For our purposes we shall consider instead the group $\mathrm{SU}(2,1)$ of matrices that are unitary with respect to H and have determinant 1. Therefore $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{I,\omega I,\omega^2 I\}$, where ω is a non real cube root of unity, and so $\mathrm{SU}(2,1)$ is a 3-fold covering of $\mathrm{PU}(2,1)$. This is analogous to the well known fact that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$. Since $\mathrm{SU}(2,1)$ comprises 3×3 matrices rather than 2×2 matrices we obtain a triple cover rather than a double cover.

The usual trichotomy which classifies isometries of real hyperbolic spaces also holds here. Namely:

(i) an isometry is *loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$;

(ii) an isometry is *parabolic* if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$;

(iii) an isometry is *elliptic* if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

In Section 2.5 below we will give another geometrical interpretation of this trichotomy.

2.4 The Heisenberg group and the boundary of $\mathbf{H}^2_{\mathbb{C}}$

Each unipotent, upper triangular matrix in $\mathrm{SU}(2,1)$ (with respect to the second Hermitian form) has the form

$$T_{(\zeta,v)} = \begin{bmatrix} 1 & -\sqrt{2}\,\overline{\zeta} & -|\zeta|^2 + iv \\ 0 & 1 & \sqrt{2}\zeta \\ 0 & 0 & 1 \end{bmatrix}$$

where $\zeta \in \mathbb{C}$ and $v \in \mathbb{R}$. There is a natural map $\phi : \mathbb{C}\times\mathbb{R}$ given by $\phi(\zeta, v) = T_{(\zeta,v)}$. The set of unipotent, upper triangular matrices in $\mathrm{SU}(2,1)$ is group under matrix multiplication. In order to make ϕ a homomorphism, we give $\mathbb{C}\times\mathbb{R}$ the following group operation:

$$(\zeta, v)\cdot(\zeta', v') = (\zeta+\zeta', v+v'+2\Im(\zeta\overline{\zeta'}),$$

that is the group structure of the *Heisenberg group* $\mathfrak{N}$.

Given a finite point z of $\partial\mathbf{H}^2_{\mathbb{C}}$ there is a unique unipotent, upper triangular matrix in $\mathrm{SU}(2,1)$ taking $o = (0,0)$ to z. Therefore we may identify $\partial\mathbf{H}^2_{\mathbb{C}}$ with the one point compactification of the Heisenberg group. (This generalises the well known fact that the boundary of hyperbolic three-space is the one point compactification of $\mathbb{C}$.)

The identification is done as follows. If z is a finite point of the boundary (that is any point besides ∞), then its standard lift is

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix}, \quad 2\Re(z_1) + |z_2|^2 = 0.$$

We write $\zeta = z_2/\sqrt{2} \in \mathbb{C}$ and this condition becomes $2\Re(z_1) = -2|\zeta|^2$. Hence we may write $z_1 = -|\zeta|^2 + iv$ for $v \in \mathbb{R}$. That is for $(\zeta, v) \in \mathfrak{N}$:

$$\mathbf{z} = \begin{bmatrix} -|\zeta|^2 + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}$$

The Heisenberg group is not abelian but is 2-step nilpotent. Therefore any point in $\mathfrak{N}$ of the form $(0, t)$ is central and the commutator of any two elements lies in the centre.

Geometrically, we think of the $\mathbb{C}$ factor of $\mathfrak{N}$ as being horizontal and the $\mathbb{R}$ factor as being vertical. We refer to $T(\zeta, v)$ as *Heisenberg translation* by (ζ, v). A Heisenberg translation by $(0, t)$ is called *vertical translation* by t. It is easy to see the Heisenberg translations are ordinary translations in the horizontal direction and shears in the vertical direction. The fact that $\mathfrak{N}$ is nilpotent means that translating around a horizontal square gives a vertical translation, rather like going up a spiral staircase. We define *vertical projection* $\Pi : \mathfrak{N} \to \mathbb{C}$ to be the map $\Pi(\zeta, v) = \zeta$.

The Heisenberg norm is given by

$$|(\zeta, v)| = \Big| |\zeta|^2 - iv \Big|^{1/2}.$$

This gives rise to a metric, the Cygan metric, on $\mathfrak{N}$ by

$$\rho_0((\zeta_1, v_1), (\zeta_2, v_2)) = |(\zeta_1, v_1)^{-1} \cdot (\zeta_2, v_2)|.$$

2.5 Geodesic submanifolds

There are no totally geodesic, real hypersurfaces of $\mathbf{H}^2_{\mathbb{C}}$, but there are are two kinds of totally geodesic 2-dimensional subspaces of complex hyperbolic space, (see Section 3.1.11 of [27]). Namely:

(i) *complex lines* L, which have constant curvature -1, and

(ii) totally real *Lagrangian planes* R, which have constant curvature $-1/4$.

Every complex line L is the image under some element of $\mathrm{SU}(2, 1)$ of the complex line L_1 where the first coordinate is zero:

$$L_1 = \Big\{ (z_1, z_2)^t \in \mathbf{H}^2_{\mathbb{C}} \ : \ z_1 = 0 \Big\}.$$

The subgroup of $\mathrm{SU}(2,1)$ stabilising L_1 is thus the group of block diagonal matrices $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1,1)) < \mathrm{SU}(2,1)$. The stabiliser of every other complex line is conjugate to this subgroup. The restriction of the Bergman metric to L_1 is just the Poincaré metric on the unit disc with curvature -1.

Every Lagrangian plane is the image under some element of $\mathrm{SU}(2,1)$ of the *standard real Lagrangian plane* $R_{\mathbb{R}}$ where both coordinates are real:

$$R_{\mathbb{R}} = \Big\{ (z_1, z_2)^t \in \mathbf{H}^2_{\mathbb{C}} \ : \ \Im(z_1) = \Im(z_2) = 0 \Big\}.$$

The subgroup of $\mathrm{SU}(2,1)$ stabilising $R_{\mathbb{R}}$ comprises matrices with real entries, that is $\mathrm{SO}(2,1) < \mathrm{SU}(2,1)$. The stabiliser of every other Lagrangian plane is conjugate to this subgroup. The restriction of the Bergman metric to $R_{\mathbb{R}}$ is the Klein-Beltrami metric on the unit disc with curvature $-1/4$.

We finally define two classes of topological circles, which form the boundaries of complex lines and Lagrangian planes respectively:

(i) the boundary of a complex line is called a $\mathbb{C}$-*circle* and

(ii) the boundary of a Lagrangian plane is called an $\mathbb{R}$-*circle*.

The complex conjugation map $\iota_{\mathbb{R}} : (z_1, z_2) \longmapsto (\overline{z}_1, \overline{z}_2)$ is an involution of $\mathbf{H}^2_{\mathbb{C}}$ fixing the standard real Lagrangian plane $R_{\mathbb{R}}$. It too is an isometry. Indeed any anti-holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ may be written as $\iota_{\mathbb{R}}$ followed by some element of $\mathrm{PU}(2,1)$. Any Lagrangian plane may be written as $R = B(R_{\mathbb{R}})$ for some $B \in \mathrm{SU}(2,1)$ and so $\iota = B\iota_{\mathbb{R}}B^{-1}$ is an anti-holomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ fixing R.

Falbel and Zocca [20] have used involutions fixing Lagrangian planes to give the following characterisation of elements of $\mathrm{SU}(2,1)$:

Theorem 2.4 (Falbel and Zocca [20]) *Any element A of $\mathrm{SU}(2,1)$ may be written as $A = \iota_1 \circ \iota_0$ where ι_0 and ι_1 are involutions fixing Lagrangian planes R_0 and R_1 respectively. Moreover*

(i) $A = \iota_1 \circ \iota_0$ is loxodromic if and only if R_0 and R_1 are disjoint;

(ii) $A = \iota_1 \circ \iota_0$ is parabolic if and only if R_0 and R_1 intersect in exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$;

(iii) $A = \iota_1 \circ \iota_0$ is elliptic if and only if R_0 and R_1 intersect in at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

2.6 Cartan's angular invariant

Let z_1, z_2, z_3 be three distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ with lifts $\mathbf{z}_1$, $\mathbf{z}_2$ and $\mathbf{z}_3$. *Cartan's angular invariant* [9] is defined as follows:

$$\mathbb{A}\,(z_1, z_2, z_3) = \arg(-\langle \mathbf{z}_1, \mathbf{z}_2\rangle\langle \mathbf{z}_2, \mathbf{z}_3\rangle\langle \mathbf{z}_3, \mathbf{z}_1\rangle).$$

The angular invariant is independent of the chosen lifts $\mathbf{z}_j$ of the points z_j. It is clear that applying an element of $\mathrm{SU}(2,1)$ to our triple of points does not change the Cartan invariant. The converse is also true.

Proposition 2.5 (Goldman, Theorem 7.1.1 of [27]) *Let z_1, z_2, z_3 and z_1', z_2', z_3' be triples of distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Then $\mathbb{A}(z_1, z_2, z_3) = \mathbb{A}(z_1', z_2', z_3')$ if and only if there exists an $A \in \mathrm{SU}(2,1)$ so that $A(z_j) = z_j'$ for $j = 1, 2, 3$. Moreover, A is unique unless the three points lie on a complex line.*

The properties of $\mathbb{A}$ may be found in Section 7.1 of [27]. In the next proposition we highlight some of them, see Corollary 7.1.3 and Theorem 7.1.4 on pages 213-4 of [27].

Proposition 2.6 (Cartan [9]) *Let z_1, z_2, z_3 be three distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ and let $\mathbb{A} = \mathbb{A}(z_1, z_2, z_3)$ be their angular invariant. Then,*

(i) $\mathbb{A} \in [-\pi/2, \pi/2]$;

(ii) $\mathbb{A} \pm \pi/2$ *if and only if z_1, z_2 and z_3 all lie on a complex line;*

(iii) $\mathbb{A} = 0$ *if and only if z_1, z_2 and z_3 all lie on a Lagrangian plane.*

Geometrically, the angular invariant $\mathbb{A}$ has the following interpretation; see Section 7.1.2 of [27]. Let L_{12} be the complex line containing z_1 and z_2 and let Π_{12} denote orthogonal projection onto L_{12}. The Bergman metric restricted to L_{12} is just the Poincaré metric with curvature -1. Consider the hyperbolic triangle Δ in L_{12} with vertices z_1, z_2 and $\Pi_{12}(z_3)$. Then the angular invariant $\mathbb{A}(z_1, z_2, z_3)$ is half the signed Poincaré area of Δ. That is, the area of Δ is $|2\mathbb{A}|$. The sign of $\mathbb{A}$ is determined by the order one meets the vertices of Δ when going around $\partial\Delta$ in a positive sense with respect to the natural orientation of L_{12}. If one meets the vertices in a cyclic permutation of z_1, z_2, $\Pi_{12}(z_3)$ then $\mathbb{A} > 0$; if one meets them in a cyclic permutation of z_2, z_1, $\Pi_{12}(z_3)$ then $\mathbb{A} < 0$. In the case where $\Pi_{12}(z_3)$ lies on the geodesic joining z_1 and z_2 then the triangle is degenerate and has area zero. In the case where z_3 lies on ∂L_{12} then Δ is an ideal triangle and has area π.

2.7 The Korányi-Reimann cross-ratio

Cross-ratios were generalised to complex hyperbolic space by Korányi and Reimann [35]. Following their notation, we suppose that z_1, z_2, z_3, z_4 are four distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$. Let $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ and $\mathbf{z}_4$ be corresponding lifts in $V_0 \subset \mathbb{C}^{2,1}$. The *complex cross-ratio* of our four points is defined to be

$$\mathbb{X} = [z_1, z_2, z_3, z_4] = \frac{\langle \mathbf{z}_3, \mathbf{z}_1 \rangle \langle \mathbf{z}_4, \mathbf{z}_2 \rangle}{\langle \mathbf{z}_4, \mathbf{z}_1 \rangle \langle \mathbf{z}_3, \mathbf{z}_2 \rangle}.$$

We note that $\mathbb{X}$ is invariant under $\mathrm{SU}(2,1)$ and independent of the chosen lifts. More properties of the complex cross-ratio may be found in Section 7.2 of [27].

By choosing different orderings of our four points we may define other cross-ratios. There are some symmetries associated to certain permutations, see Property 5 on page 225 of [27]. After taking these into account, there are three cross ratios that remain. Given four distinct points $z_1, \ldots, z_4 \in \partial\mathbf{H}^2_{\mathbb{C}}$, we define

$$\mathbb{X}_1 = [z_1, z_2, z_3, z_4], \quad \mathbb{X}_2 = [z_1, z_3, z_2, z_4], \quad \mathbb{X}_3 = [z_2, z_3, z_1, z_4]. \tag{1}$$

These three cross-ratios determine the quadruple of points up to $\mathrm{SU}(2,1)$ equivalence; see Falbel [16] or Parker and Platis [49].

Proposition 2.7 (Falbel [16]) *Let $z_1, \ldots, z_4$ be distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ with cross ratios $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ given by (1). If $z_1', \ldots, z_4'$ is another set be distinct points of $\partial\mathbf{H}^2_{\mathbb{C}}$ so that*

$$\begin{aligned} \mathbb{X}_1 &= [z_1, z_2, z_3, z_4] = [z_1', z_2', z_3', z_4'], \\ \mathbb{X}_2 &= [z_1, z_3, z_2, z_4] = [z_1', z_3', z_2', z_4'], \\ \mathbb{X}_3 &= [z_2, z_3, z_1, z_4] = [z_2', z_3', z_1', z_4']. \end{aligned}$$

Then there exists $A \in \mathrm{SU}(2,1)$ so that $A(z_j) = z_j'$ for $j = 1, 2, 3, 4$.

In [16] Falbel has given a general setting for cross-ratios that includes both Korányi-Reimann cross ratios and the standard real hyperbolic cross-ratio. The normalisation (1) is somewhat different than his. The three cross-ratios satisfy two real equations; in Falbel's normalisation, the analogous relations are given in Proposition 2.3 of [16]. In his general setting there are six cross-ratios that lie on a complex algebraic variety in $\mathbb{C}^6$. Our cross-ratios correspond to the fixed locus of an antiholomorphic involution on this variety.

Proposition 2.8 (Parker and Platis [49]) *Let z_1, z_2, z_3, z_4 be any four distinct points in $\partial\mathbf{H}^2_{\mathbb{C}}$. Let $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ be defined by (1). Then*

$$\begin{aligned} |\mathbb{X}_2| &= |\mathbb{X}_1|\,|\mathbb{X}_3|, && (2)\\ 2|\mathbb{X}_1|^2\Re(\mathbb{X}_3) &= |\mathbb{X}_1|^2+|\mathbb{X}_2|^2+1-2\Re(\mathbb{X}_1+\mathbb{X}_2). && (3)\end{aligned}$$

The set of points $\mathfrak{X}$ consisting of triples $(\mathbb{X}_1,\mathbb{X}_2,\mathbb{X}_3)$ is thus an algebraic variety which we call *cross-ratio variety*.

The cross-ratio variety $\mathfrak{X}$ defined in Proposition 2.8 appears naturally in the study of the space of configurations of four points in the unit sphere S^3. In the classical case, it is well known that a configuration of four points on the Riemann sphere is determined up to Möbius equivalence by their cross-ratio and the set F of equivalence classes of configurations is biholomorphic to $\mathbb{C}-\{0,1\}$ which may be identified to the set of cross-ratios of four pairwise distinct points on $\mathbb{CP}^1$ (see section 4.4 of [5]). In the complex hyperbolic setting, things are more complicated. Denote by $\mathcal{C}$ the set of configurations of four points in S^3. The group of holomorphic isometries of $\mathbf{H}^2_{\mathbb{C}}$ acts naturally on $\mathcal{C}$. Denote by $\mathcal{F}$ the quotient of $\mathcal{C}$ by this action. In [16], Falbel proved the following.

Proposition 2.9 (Falbel [16]) *There exists a CR map $\pi:\mathcal{F}\to\mathbb{C}^3$ such that its image is $\mathfrak{X}$.*

Now, there are plenty of analytical and geometrical structures on large subsets of $\mathcal{F}$, all inherited from the natural CR structure of the Heisenberg group. Namely, (see [19]) there exists a complex structure and a (singular) CR structure of codimension 2. To transfer these results to $\mathfrak{X}$, Falbel and Platis strengthened Proposition 2.9 by proving that the map π is a CR-embedding. Thus they prove the following.

Theorem 2.10 (Falbel and Platis [19]) *Let $\mathfrak{X}$ be the cross-ratio variety. Then, except some subsets of measure zero, $\mathfrak{X}$ is*

(i) a 2-complex manifold and

(ii) admits a singular CR structure of codimension 2.

In [50] Parker and Platis explored the topology of the cross-ratio variety by defining global coordinates on $\mathfrak{X}$ using only geometrical tools.

3 Representations of surface groups

In what follows we fix an oriented topological surface Σ of genus g with p punctures; its Euler characteristic is thus $\chi = \chi(\Sigma) = 2 - 2g - p$. We suppose that $\chi < 0$. We denote by $\pi_1 = \pi_1(\Sigma)$ the fundamental group of Σ. A specific choice of generators for π_1 is called a *marking*. The collection of marked representations of π_1 into a Lie group G up to conjugation will be denoted by $\mathrm{Hom}(\pi_1, G)/G$. We shall consider $\mathrm{Hom}(\pi_1, G)/G$ endowed with the compact-open topology; this will enable us to make sense of what it means for two representations to be close. We remark that in the cases we consider, the compact-open topology is equivalent to the l^2-topology on the relevant matrix group.

Our main interest is in the case where $G = \mathrm{SU}(2,1)$, that is representation variety consisting of marked representations up to conjugation into the group of holomorphic isometries of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$. In the following section we review in brief the classical cases of spaces of marked representations of π_1 into G when G is $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$. These spaces are the predecessors of the space we study here.

3.1 The motivation from real hyperbolic geometry

It is well known that if $\rho : \pi_1 \longrightarrow \mathrm{SL}(2,\mathbb{R})$ is a discrete and faithful representation of π_1, then $\rho(\pi_1)$ is called *Fuchsian*. In this case ρ is necessarily geometrically finite. Moreover, $\rho(\pi_1)$ is necessarily totally loxodromic when $p = 0$. If $p > 0$ then this condition would be replaced with type-preserving, which requires that an element of $\rho(\pi_1)$ is parabolic if and only if it represents a peripheral curve. The group $\mathrm{SL}(2,\mathbb{R})$ is a double cover of the group of orientation preserving isometries of the hyperbolic plane. The quotient of the hyperbolic plane by $\rho(\pi_1)$ naturally corresponds to a hyperbolic (as well as to a complex) structure on Σ.

Definition 3.1 *The collection of distinct, marked Fuchsian representations, up to conjugacy within* $\mathrm{SL}(2,\mathbb{R})$*, is the Teichmüller space of* Σ*, denoted* $\mathcal{T} = \mathcal{T}(\Sigma) \subset \mathrm{Hom}(\pi_1, \mathrm{SL}(2,\mathbb{R}))/\mathrm{SL}(2,\mathbb{R})$.

Among its many of properties, Teichmüller space is:

- topologically a ball of real dimension $6g - 6 - 2p$,
- a complex Banach manifold, equipped with a Kähler metric (the Weil-Petersson metric) of negative holomorphic sectional curvature.

We now consider representations of π_1 to $\mathrm{SL}(2,\mathbb{C})$. We again require such a representation ρ to be discrete, faithful, geometrically finite and, if $p = 0$ (respectively $p > 0$) totally loxodromic (respectively type preserving). We call these representations *quasi-Fuchsian*.

Definition 3.2 *The collection of distinct, marked quasi-Fuchsian representations, up to conjugation in* $\mathrm{SL}(2,\mathbb{C})$ *is called (real hyperbolic) quasi-Fuchsian space and is denoted by* $\mathcal{Q}_{\mathbb{R}} = \mathcal{Q}_{\mathbb{R}}(\Sigma) \subset \mathrm{Hom}(\pi_1,\ \mathrm{SL}(2,\mathbb{C}))/\mathrm{SL}(2,\mathbb{C})$.

If ρ is a quasi-Fuchsian representation, then it corresponds to a three dimensional hyperbolic structure on an interval bundle over Σ. Real hyperbolic quasi-Fuchsian space $\mathcal{Q}_{\mathbb{R}}(\Sigma)$:

- may be identified with the product of two copies of Teichmüller space according to the Simultaneous Uniformization Theorem of Bers [6],
- it is a complex manifold of dimension $6g - 6 - 2p$ and it is endowed with a hyper-Kähler metric whose induced complex symplectic form is the complexification of the Weil-Petersson symplectic form on $\mathcal{T}(\Sigma)$; see [52].

3.2 Complex hyperbolic quasi-Fuchsian space

Motivated by these two examples, one may consider representations of π_1 into $\mathrm{SU}(2,1)$ up to conjugation, that is $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$. A definition consistent with the ones in the classical cases would then be the following.

Definition 3.3 *A representation in* $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ *is said to be complex hyperbolic quasi-Fuchsian if it is*

- *discrete,*
- *faithful,*
- *geometrically finite and*
- *type-preserving.*

We remark that in the case where $p = 0$ we may replace “geometrically finite” with “convex-cocompact” and we may replace “type-preserving” with “totally loxodromic”.

Since the group $\mathrm{SU}(2,1)$ is a triple cover of the holomorphic isometry group of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ it turns out that such a representation corresponds to a complex hyperbolic structure on a disc bundle over Σ.

There are two matters to clarify in Definition 3.3. The first has to do with the geometrical finiteness of a representation. The notion of geometrical finiteness in spaces with variable negative curvature generalises the one in spaces with constant negative curvature and has been studied by Bowditch in [8]. (We keep Bowditch's labels, in particular there is no condition F3.) These definitions require the extensions to complex hyperbolic space of several familiar notion's from the theory of Kleinian groups. We refer to [8] for details of how this extension takes place. In particular, $\mathrm{core}(M)$ is the quotient by Γ of the convex hull of the limit set Λ, that is the *convex core*; $\mathrm{thin}_\epsilon(M)$ is the part of M where the injectivity radius is less than ϵ, that is the *ϵ-thin part* and $\mathrm{thick}_\epsilon(M)$ is its complement. In [8] Bowditch explains that there is no condition F3 because that would involve finite sided fundamental polyhedra.

Theorem 3.4 (Bowditch [8]) *Let Γ be a discrete subgroup of $\mathrm{SU}(2,1)$. Let $\Lambda \subset \partial\mathbf{H}^2_{\mathbb{C}}$ be the limit set of Γ and let $\Omega = \partial\mathbf{H}^2_{\mathbb{C}} - \Lambda$ be the domain of discontinuity of Γ. Let $M_C(\Gamma)$ denote the orbifold $(\mathbf{H}^2_{\mathbb{C}} \cup \Omega)/\Gamma$. Then the following conditions are equivalent and any group satisfying one of them will be called geometrically finite:*

- *F1. $M_C(\Gamma)$ has only finitely many topological ends, each of which is a parabolic end.*
- *F2. Λ consists only of conical limit points and bounded parabolic fixed points.*
- *F4. There exists $\epsilon > 0$ so that $\mathrm{core}(M) \cap \mathrm{thick}_\epsilon(M)$ is compact. Here ϵ is chosen small enough so that $\mathrm{thin}_\epsilon(M)$ is the union of cusps and Margulis tubes.*
- *F5. There is a bound on the orders of every finite subgroup of Γ and there exists $\eta > 0$ so that the η neighbourhood of $\mathrm{core}(M)$ has finite volume.*

The second issue to clarify in Definition 3.3 concerns type-preserving representations. As for Fuchsian groups, a type-preserving representation is automatically discrete. This follows from the following theorem of Chen and Greenberg. In spite of this, we include the hypothesis of discreteness first because it reinforces the connection with the classical definition and, secondly, because our usual method of showing that a representation is quasi-Fuchsian is to show that it is discrete. In particular we usually construct a fundamental polyhedron and use Poincaré's polyhedron theorem to verify the other criteria.

Theorem 3.5 (Chen and Greenberg [10]) *Let* Γ *be a discrete, non-elementary subgroup of* $\mathrm{SU}(2,1)$*. If the identity is not an accumulation point of elliptic elements of* Γ *then* Γ *is discrete.*

Note that in Chen and Greenberg's statement, Corollary 4.5.3 of [10], they suppose that Γ has more than one limit point and the lowest dimensional, Γ-invariant, totally-geodesic subspace of $\mathbf{H}^2_{\mathbb{C}}$ has even dimensions. Since the only odd dimensional, totally geodesic submanifolds of $\mathbf{H}^2_{\mathbb{C}}$ are geodesics, in our case this means that Γ does not fix a point of $\overline{\mathbf{H}^2_{\mathbb{C}}}$ and does not leave a geodesic invariant. Hence Γ is non-elementary. (Our statement does not hold for $\mathrm{SU}(3,1)$, although Chen and Greenberg's does. This is because there are totally loxodromic, non-elementary, non-discrete subgroups of $\mathrm{SO}(3,1) < \mathrm{SU}(3,1)$ preserving a copy of hyperbolic 3-space.)

This contrasts with the case of representations to $\mathrm{SL}(2,\mathbb{C})$. In our definition of complex hyperbolic quasi-Fuchsian we have included the conditions that such a representation should be both discrete and totally loxodromic.

Definition 3.6 *The space of all marked complex hyperbolic quasi-Fuchsian representations, up to conjugacy, will be called* complex hyperbolic quasi-Fuchsian space

$$\mathcal{Q}_{\mathbb{C}} = \mathcal{Q}_{\mathbb{C}}(\Sigma) \subset \mathrm{Hom}(\pi_1,\, \mathrm{SU}(2,1))/\mathrm{SU}(2,1).$$

3.3 Fuchsian representations

It is reasonable to start our study of complex hyperbolic quasi-Fuchsian space by considering the Fuchsian representations inside this space. There are two ways to make a Fuchsian representation act on $\mathbf{H}^2_{\mathbb{C}}$. These correspond to the two types of totally geodesic, isometric embeddings of the hyperbolic plane into $\mathbf{H}^2_{\mathbb{C}}$ as we have seen in section 2.5, namely, totally real Lagrangian planes and complex lines.

(i) If a discrete, faithful representation ρ is conjugate to a representation of π_1 into $\mathrm{SO}(2,1) < \mathrm{SU}(2,1)$ then it preserves a Lagrangian plane and is called $\mathbb{R}$-*Fuchsian*.

(ii) If a discrete, faithful representation ρ is conjugate to a representation of π_1 into $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1,1)) < \mathrm{SU}(2,1)$ then it preserves a complex line and is called $\mathbb{C}$-*Fuchsian*.

We shall see later that $\mathbb{C}$-Fuchsian representations are distinguished in the space $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.

3.4 Toledo invariant

Let G be $\mathrm{PU}(2,1)$ or $\mathrm{SU}(2,1)$ and $\rho : \pi_1 \longrightarrow G$ be a representation. There is an important invariant associated to ρ called the *Toledo invariant* [65], which is defined as follows. Let $\widetilde{f} : \widetilde{\Sigma} \to \mathbf{H}^2_{\mathbb{C}}$ be a ρ-equivariant smooth mapping of the universal cover $\widetilde{\Sigma}$ of Σ. Then the Toledo invariant $\tau(\rho, \widetilde{f})$ is defined as the (normalised) integral of the pull-back of the Kähler form ω on $\mathbf{H}^2_{\mathbb{C}}$. Namely,

$$\tau(\rho, \widetilde{f}) = \frac{1}{2\pi}\int_{\Omega} \widetilde{f}^*\omega$$

where Ω is a fundamental domain for the action of π_1 on $\widetilde{\Sigma}$. The main properties of the Toledo invariant are the following.

Proposition 3.7 *(i) τ is independent of $\widetilde{f}$ and varies continuously with ρ,*

(ii) $\chi \leq \tau(\rho) \leq -\chi$, see [12, 33].

Geometrically, one may think of the Toledo invariant as follows; see Section 7.1.4 of [27]. Take a triangulation of Σ and consider the lift of this triangulation to the equivariant embedding $\widetilde{f}(\widetilde{\Sigma})$ of the universal cover of Σ constructed above. The result is a triangle Δ in $\mathbf{H}^2_{\mathbb{C}}$ with vertices z_1, z_2, z_3. Suppose that the edges of Δ are the oriented geodesic arcs from z_j to z_{j+1} (with indices taken mod 3). Since ω is an exact form on $\mathbf{H}^2_{\mathbb{C}}$ its integral over Δ only depends on the boundary, therefore the value of the integral is independent of the choice of triangle filling the edges. Let L_{12} be the complex line containing z_1 and z_2 and let Π_{12} denote the orthogonal projection to L_{12}. Then the integral of ω over Δ is the signed area (with the Poincaré metric) of the triangle in L_{12} with vertices z_1, z_2 and $\Pi_{12}(z_3)$. This is called *Toledo's cocycle* [65]. It is easy to see that this is an extension of Cartan's angular invariant to the case of triangles whose vertices do not necessarily lie on $\partial\mathbf{H}^2_{\mathbb{C}}$; compare Section 2.6. In order to find the Toledo invariant, one repeats this construction over all triangles in the triangulation of Σ and takes the sum. The result is $2\pi\tau(\rho)$. For each triangle, the maximum value of the Toledo cocycle is the hyperbolic area and this occurs if the boundary lies in a complex line. The only way for the Toledo invariant to take its maximal value is if all the triangles lie in the same complex line and are consistently oriented, that is ρ is $\mathbb{C}$-Fuchsian. In which case (when the triangles are all positively oriented) we have $\tau(\rho) = \mathcal{A}/2\pi = -\chi(\Sigma)$, where $\mathcal{A}$ is the area of Σ with a Poincaré metric. Likewise, applying an antiholomorphic isometry of $\mathbf{H}^2_{\mathbb{C}}$ changes the orientation of the triangle Δ and so changes the sign of the Toledo cocycle. Thus, if all triangles are negatively oriented we get $\tau(\rho) = -\mathcal{A}/2\pi = \chi(\Sigma)$.

If ρ is $\mathbb{R}$-Fuchsian, then it is invariant under $\iota_{\mathbb{R}}$ and so $\tau(\rho) = -\tau(\rho)$. Hence $\tau(\rho) = 0$. This gives a sketch proof of the following result. Note that it may be the case that $\tau(\rho) = 0$ and for ρ to not be $\mathbb{R}$-Fuchsian.

Proposition 3.8 *Suppose that $\rho \in \mathcal{Q}_{\mathbb{C}}(\Sigma)$. Then,*

(i) ρ is $\mathbb{C}$-Fuchsian if and only if $|\tau(\rho)| = -\chi$, see [65],

(ii) if ρ is $\mathbb{R}$-Fuchsian then $\tau(\rho) = 0$, see [28].

3.5 Fundamental domains

We have already mentioned that unlike the case of constant curvature there exist no totally geodesic hypersurfaces in complex hyperbolic space. Before attempting to construct a fundamental domain in $\mathbf{H}^2_{\mathbb{C}}$ we must choose the class of real hypersurfaces containing its faces. Moreover, since these faces are not totally geodesic they may intersect in complicated ways. Therefore, constructing fundamental domains in $\mathbf{H}^2_{\mathbb{C}}$ is quite a challenge. In sections 3.6 and 3.7 below we discuss two classes of hypersurfaces, namely bisectors and packs. Fundamental domains whose faces are contained in bisectors have been widely studied, in particular, this is the case for the construction of Dirichlet domains. The idea goes back to Giraud and was developed further by Mostow and Goldman (see [27] and the references therein). In section 4 we shall see how such domains were constructed by Goldman and Parker in [29] and Gusevskii and Parker in [33]. On the other hand, in order to build fundamental domains, Falbel and Zocca used $\mathbb{C}$-spheres [20] and Schwartz used $\mathbb{R}$-spheres in [58] (for the relationship between $\mathbb{C}$-spheres and $\mathbb{R}$-spheres see [18]). Packs have been used by Will [66] and by Parker and Platis [48].

3.6 Bisectors

A bisector $\mathcal{B}$ is the locus of points equidistant from a given pair of points z_1 and z_2 in $\mathbf{H}^2_{\mathbb{C}}$. In other words:

$$\mathcal{B} = \mathcal{B}(z_1, z_2) = \Big\{ z \in \mathbf{H}^2_{\mathbb{C}} \ : \ \rho(z, z_1) = \rho(z, z_2) \Big\}.$$

The corresponding construction in real hyperbolic geometry gives a hyperbolic hyperplane, which is totally geodesic. We have already seen that there are no totally geodesic real hypersurfaces in complex hyperbolic space. Therefore $\mathcal{B}$ is not totally geodesic. However, it is as close as it can be to being totally geodesic, in that it is foliated by totally geodesic subspaces in two distinct ways. For an extensive treatment of bisectors readers should see [27].

Let z_1 and z_2 be any two points of $\mathbf{H}^2_{\mathbb{C}}$. Then z_1 and z_2 lie in a unique complex line L, which we call the the *complex spine* of $\mathcal{B}$. (Note that the complex spine is often denoted Σ, but this conflicts with our use of Σ to denote a surface.) The restriction of the Bergman metric to L is the Poincaré metric. Thus the points in L equidistant from z_1 and z_2 lie on a Poincaré geodesic in L, which we call the *spine* of $\mathcal{B}$ and denote by σ. The endpoints of the spine are called the *vertices* of $\mathcal{B}$. Goldman shows (page 154 of [27]) that a bisector is completely determined by its vertices and so $\mathrm{SU}(2,1)$ acts transitively on the set of all bisectors.

Following earlier work of Giraud, Mostow describes a foliation of bisectors by complex lines as follows (see also Section 5.1.2 of [27]):

Theorem 3.9 (Mostow [46]) *Let $\mathcal{B}$ be a bisector and let σ and L denote its spine and complex spine respectively. Let Π_L denote orthogonal projection onto L and for each $s \in \sigma$ let L_s be the complex line so that $\Pi_L(L_s) = s$. Then*

$$\mathcal{B} = \bigcup_{s \in \sigma} L_s = \bigcup_{s \in \sigma} \Pi_L^{-1}(s).$$

Goldman describes a second foliation of bisectors, this time by Lagrangian planes.

Theorem 3.10 (Goldman [27]) *Let $\mathcal{B}$ be a bisector with spine σ. Then $\mathcal{B}$ is the union of all Lagrangian planes containing σ.*

The complex lines L_s defined in Theorem 3.9 are called the *slices* of $\mathcal{B}$. The Lagrangian planes defined in Theorem 3.10 are called the *meridians* of $\mathcal{B}$. Together the slices and the meridians of a bisector give *geographical coordinates*. The reason for this name is that the boundary of a bisector in $\partial\mathbf{H}^2_{\mathbb{C}}$ is topologically a sphere, called by Mostow a *spinal sphere*. The boundaries of the slices and meridians are $\mathbb{C}$-circles and $\mathbb{R}$-circles that foliate the spinal sphere in a manner analogous to lines of latitude and lines of longitude, respectively, on the earth.

3.7 Packs

Bisectors are rather badly adapted to $\mathbb{R}$-Fuchsian representations. When considering representations close to an $\mathbb{R}$-Fuchsian representation it is more convenient to use a different class of hypersurfaces, called *packs*. There is a natural partial duality, resembling mirror symmetry, in complex hyperbolic space between complex objects (such as complex lines) and totally real objects (such as Lagrangian planes), see the discussion in the introduction to [18]. From this point of view, packs are the dual objects to bisectors. Packs were introduced by Will in [66] and in their general form by Parker and Platis in [48].

Let A be a loxodromic map in $\mathrm{SU}(2,1)$. Then, it has two fixed points in $\partial\mathbf{H}^2_{\mathbb{C}}$. Moreover, there exists a complex number $\lambda = l + i\theta \in \mathbb{R}_+ \times (-\pi, \pi]$ (the *complex length of* A) such that the eigenvalues of A are e^{λ}, $e^{\overline{\lambda}-\lambda}$, $e^{-\overline{\lambda}}$. For any $x \in \mathbb{R}$ define A^x to be the element of $\mathrm{SU}(2,1)$ which has the same eigenvectors as A, but its eigenvalues are the eigenvalues of A raised to the xth power. Hence we immediately see that A^x is a loxodromic element of $\mathrm{SU}(2,1)$ for all $x \in \mathbb{R} - \{0\}$ and $A^0 = I$. The following Proposition, see [48] holds.

Proposition 3.11 *Let R_0 and R_1 be disjoint Lagrangian planes in $\mathbf{H}^2_{\mathbb{C}}$ and let ι_0 and ι_1 be the respective inversions. Consider $A = \iota_1\iota_0$ (which is loxodromic map by Theorem 2.4) and its powers A^x for each $x \in \mathbb{R}$. Then:*

(i) ι_x defined by $A^x = \iota_x\iota_0$ is inversion in a Lagrangian plane $R_x = A^{x/2}(R_0)$.

(ii) R_x intersects the complex axis L_A of A orthogonally in a geodesic γ_x.

(iii) The geodesics γ_x are the leaves of a foliation of L_A.

(iv) For each $x \neq y \in \mathbb{R}$, R_x and R_y are disjoint.

Definition 3.12 *Given disjoint Lagrangian planes R_0 and R_1, then for each $x \in \mathbb{R}$ let R_x be the Lagrangian plane constructed in Proposition 3.11. Define*

$$\mathcal{P} = \mathcal{P}(R_0, R_1) = \bigcup_{x \in \mathbb{R}} R_x = \bigcup_{x \in \mathbb{R}} A^{x/2}(R_0).$$

Then $\mathcal{P}$ is a real analytic 3-submanifold which we call the pack determined by R_0 and R_1. We call the axis of $C = \iota_1 \iota_0$ the spine of $\mathcal{P}$ and the Lagrangian planes R_x for $x \in \mathbb{R}$ the slices of $\mathcal{P}$.

Observe that $\mathcal{P}$ contains L, the complex line containing γ, the spine of $\mathcal{P}$. The following Proposition is obvious from the construction and emphasises the similarity between bisectors and packs (compare it with Section 5.1.2 of [27]).

Proposition 3.13 *Let $\mathcal{P}$ be a pack. Then $\mathcal{P}$ is homeomorphic to a 3-ball whose boundary lies in $\partial \mathbf{H}^2_{\mathbb{C}}$. Moreover, $\mathbf{H}^2_{\mathbb{C}} - \mathcal{P}$, the complement of $\mathcal{P}$, has two components, each homeomorphic to a 4-ball.*

We remark that the boundary of $\mathcal{P}$ contains the boundary of the complex line L and is foliated by the boundaries of the Lagrangian planes R_x. Since it is also homeomorphic to a sphere, it is an example of an $\mathbb{R}$-sphere (hybrid sphere), see [18, 58].

The definition of packs associated to loxodromic maps A that preserve a Lagrangian plane (that is with $\theta = \Im(\lambda) = 0$) was given by Will. We call the resulting pack *flat*.

Proposition 3.14 (Will, Section 6.1.1 of [66]) *Suppose that the geodesic γ lies on a totally real plane R. Then the set*

$$\mathcal{P}(\gamma) = \Pi_R^{-1}(\gamma) = \bigcup_{z \in \gamma} \Pi_R^{-1}(z).$$

is the flat pack determined by the Lagrangian planes $R_0 = \Pi_R^{-1}(z_0)$ and $R_1 = \Pi_R^{-1}(z_1)$ for any distinct points z_0, $z_1 \in \gamma$. Moreover, for each $z \in \gamma$, the Lagrangian plane $\Pi_R^{-1}(z)$ is a slice of $\mathcal{P}(\gamma)$.

Let $A \in \mathrm{SU}(2,1)$ be a loxodromic map with eigenvalues e^{λ}, $e^{\overline{\lambda}-\lambda}$, $e^{-\overline{\lambda}}$ where $\lambda = l + i\theta \in \mathbb{R}_+ \times (-\pi, \pi]$. Suppose that $\mathcal{P}$ is a pack determined by A, as in Definition 3.12. We define the *curl factor* of $\mathcal{P}$ to be $\kappa(\mathcal{P}) = \theta/\lambda = \tan(\arg(\lambda))$. Note that flat packs have curl factor 0. Platis [56] proves that two packs $\mathcal{P}_1$ and $\mathcal{P}_2$ are isometric if and only if $\kappa(\mathcal{P}_1) = \kappa(\mathcal{P}_2)$. In particular, $\mathrm{SU}(2,1)$ acts transitively on the set of flat packs.

4 Punctured surfaces

In the study of complex hyperbolic quasi-Fuchsian representations of a surface group, there are qualitatively different conclusions, depending whether $p = 0$ or $p > 0$. Thus we examine each case separately. We will suppose in this section that Σ is a surface with genus g and $p > 0$ punctures having Euler characteristic $\chi = 2 - 2g - p < 0$.

4.1 Ideal triangle groups and the three times punctured sphere

The first complex hyperbolic deformation space to be completely described is that of the group generated by reflections in the sides of an ideal triangle [29, 57]. This group has an index two subgroup corresponding to the thrice punctured sphere. In the case of constant negative curvature these groups are rigid. However, variable negative curvature allows us to deform the group.

Let Σ be a three times punctured sphere and $\pi_1 = \pi_1(\Sigma)$ its fundamental group. Abstractly π_1 is a free group on two generators and we may choose the generators α and β so that loops around the three punctures are represented by α, β and $\alpha\beta$. If $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ is a type-preserving representation of π_1 then $\rho(\alpha)$, $\rho(\beta)$ and $\rho(\alpha\beta)$ are all parabolic. We consider the special case where each of these three classes is represented by a unipotent parabolic map (that is a map conjugate to a Heisenberg translation). Let u_1, u_2 and u_0 be the fixed points of the unipotent parabolic maps $A = \rho(\alpha)$, $B = \rho(\beta)$ and $AB = \rho(\alpha\beta)$. Let L_j be the complex line spanned by u_{j-1} and u_{j+1} where the indices are taken mod 3. Let $I_j \in \mathrm{SU}(2,1)$ be the complex reflection of order 2 fixing L_j. A consequence of our hypothesis that A, B and AB are unipotent is that $A = I_1 I_0$ and $B = I_0 I_1$. Thus $\rho(\pi_1) = \langle A, B\rangle$ has index 2 in $\langle I_0, I_1, I_2\rangle$. There is a one (real) dimensional space of representations ρ of π_1 with the above properties, the parameter being the angular invariant of the three fixed points $\mathbb{A}(u_0, u_1, u_2) \in [-\pi/2, \pi/2]$.

It remains to decide which of these groups are quasi-Fuchsian. This question was considered by Goldman and Parker in [29]. They partially solved the problem and gave a conjectural picture of the complete solution. This conjecture was proved by Schwartz in [57], who also gave a more conceptual proof in [60]. We restate the main result in terms of the three times punctured sphere:

Theorem 4.1 (Goldman and Parker [29], Schwartz [57]) *Let Σ be a three times punctured sphere with fundamental group π_1. Let $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$*

be a representation of π_1 so that the three boundary components are represented by unipotent parabolic maps with fixed points u_0, u_1 and u_2. Let $\mathbb{A} = \mathbb{A}(u_0, u_1, u_2)$ be the angular invariant of these fixed points.

(i) If $-\sqrt{125/3} < \tan(\mathbb{A}) < \sqrt{125/3}$ then ρ is complex hyperbolic quasi-Fuchsian.

(ii) If $\tan(\mathbb{A}) = \pm\sqrt{125/3}$ then ρ is discrete, faithful and geometrically finite and has accidental parabolics.

(iii) If $\sqrt{125/3} < |\tan(\mathbb{A})| < \infty$ then ρ is not discrete.

(iv) If $\mathbb{A} = \pm\pi/2$ then ρ is the trivial representation.

Outline of proof. The proof of (i) and (ii) involves constructing a fundamental domain for $\Gamma = \rho(\pi_1)$. The proof of (iii) follows by showing that $I_0I_1I_2$ (or equivalently $A^{-1}BAB^{-1} = (I_0I_1I_2)^2$) is elliptic of infinite order. The proof of (iv) is trivial since in this case all three lines coincide.

Proposition 4.2 (Gusevskii and Parker [32]) *Let ρ be as in Theorem 4.1 then the Toledo invariant $\tau(\rho)$ is zero.*

4.2 The modular group and punctured surface groups

Let $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ be the classical modular group. It is well known that Γ is generated by an element of order two and an element of order three whose product is parabolic. There are six possibilities for the representation depending on the types of the generators of orders 2 and 3.

If A is a complex reflection in a point p then A preserves all complex lines through p; if A is a complex reflection in a complex line L then A preserves all complex lines orthogonal to L. Given two complex reflections there is a unique complex line through their fixed point(s) or orthogonal to their fixed line(s) respectively. This line is preserved by both reflections and hence by the group they generate. For any representation of the modular group in $\mathrm{PU}(2,1)$, the order 2 generator of the modular group must be a complex reflection. Therefore, if the order 3 generator is also a complex reflection then this representation necessarily preserves a complex line and so is $\mathbb{C}$-Fuchsian. The condition that the product of the two generators is parabolic completely determines the representation up to conjugacy.

In the remaining two cases, we consider representations where the order 3 generator is not a complex reflection, that is it has distinct eigenvalues. There are two cases, namely when the order 2 generator is a complex reflection in a point or in a line. The former case was considered by Parker and Gusevskii [33] and Falbel and Koseleff [17] independently. The latter case was considered by Falbel and Parker [18]. In both cases the representation may be parametrised as follows. Let u_0 be the fixed point of the product of the generators, which is parabolic by hypothesis. Let u_1 and u_2 be the images of this parabolic fixed point under powers of the order 3 generator. We then define $\mathbb{A} = \mathbb{A}(u_0, u_1, u_2)$ to be the angular invariant of these three fixed points. It turns out that this angular invariant completely parametrises the representation. In particular, in each case the parabolic map is not unipotent, but corresponds to a screw-parabolic map with rotational part of angle $\mathbb{A}$.

The complex hyperbolic quasi-Fuchsian space of the modular group was entirely described by Falbel and Parker in [18].

Theorem 4.3 (Falbel and Parker [18]) *Let* $\rho : \mathrm{PSL}(2,\mathbb{Z}) \longrightarrow \mathrm{PU}(2,1)$ *be a complex hyperbolic representation of the modular group. Let* $\mathbb{A}$ *be the angular invariant described above. Then:*

(i) There are four rigid $\mathbb{C}$*-Fuchsian representations for which the elliptic element of order 2 and 3 are complex reflections.*

(ii) If the order 3 generator is not a complex reflection and the order 2 generator is a complex reflection in a point then ρ *is complex hyperbolic quasi-Fuchsian for all* $\mathbb{A} \in [-\pi/2, \pi/2]$*.*

(iii) If the order 3 generator is not a complex reflection and the order 2 generator is a complex reflection in a line then:

(a) If $\sqrt{15} < |\tan(\mathbb{A})| \leq \infty$ *then* ρ *is complex hyperbolic quasi-Fuchsian.*

(b) If $\tan(\mathbb{A}) = \pm\sqrt{15}$ *then* ρ *is discrete, faithful and geometrically finite and has accidental parabolics.*

(c) If $-\sqrt{15} < \tan(\mathbb{A}) < \sqrt{15}$ *then* ρ *is either not faithful or not discrete.*

In (ii) and (iii) the representation ρ *may be lifted to an isomorphic representation to* $\mathrm{SU}(2,1)$*. In (i) this cannot be done.*

Outline of proof. In (i) the four cases correspond to the different possibilities of reflection in a point or line for each of the generators.

In cases (ii) and (iii)(a) the authors construct a parametrised family of fundamental domains. In case (iii)(c) the commutator of the order two and order 3 generators is elliptic. In $\mathrm{PSL}(2,\mathbb{Z})$ this element corresponds to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

□

Proposition 4.4 (Gusevskii and Parker [33]) *Let ρ and $\mathbb{A}$ be as in Theorem 4.3 (ii). Then $\tau(\rho) = \mathbb{A}/3\pi$.*

One remarkable fact is that the groups in Theorem 4.1 (ii) and Theorem 4.3 (iii)(b) are commensurable, and they are each commensurable with an $\mathrm{SU}(2,1)$ representation of the Whitehead link complement, as proved by Schwartz [58].

Let π_1 be the fundamental group of an orbifold Σ and suppose that $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ is a representation with Toledo invariant $\tau = \tau(\rho)$. Let $\widehat{\pi}_1$ be an index d subgroup of π_1 ($\widehat{\pi}_1$ is the fundamental group of an orbifold $\widehat{\Sigma}$ which is a d-fold cover of Σ). The restriction of ρ from π_1 to $\widehat{\pi}_1$ gives a representation $\widehat{\rho} : \widehat{\pi}_1 \longrightarrow \mathrm{SU}(2,1)$. In each case the universal cover $\widetilde{\Sigma}$ is the same and so the integral defining $\tau(\widehat{\rho})$ is the same as the integral defining $\tau(\rho)$ but taken over d copies of a fundamental domain. Hence $\tau(\widehat{\rho}) = d\tau(\rho)$. Millington showed that any punctured surface group arises as a finite index subgroup of the modular group:

Proposition 4.5 (Millington [43]) *Let $p > 0$, $g \geq 0$, $e_2 \geq 0$ and $e_3 \geq 0$ be integers and write $d = 12(g-1) + 6p + 3e_2 + 4e_3$. If $d > 0$ then there is a subgroup of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ of index d which is the fundamental group of an orbifold of genus g with p punctures removed, e_2 cone points of angle π and e_3 cone points of angle $2\pi/3$.*

We may combine the above observations to prove that for any punctured surface there exists a representation taking any value of the Toledo invariant in the interval $[\chi, -\chi]$.

Theorem 4.6 (Gusevskii and Parker [33]) *Let Σ be a surface of genus g with p punctures and Euler characteristic $\chi = 2 - 2g - p < 0$. There exists* *a continuous family of complex hyperbolic quasi-Fuchsian representations* ρ_t *of* $\pi_1(\Sigma)$ *into* $\mathrm{SU}(2,1)$ *whose Toledo invariant* $\tau(\rho_t)$ *varies between* χ *and* $-\chi$.

Sketch of proof. Let Σ be a surface of genus g with p punctures. Using Proposition 4.5, there is a type preserving representation of $\pi_1(\Sigma)$ as a subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ of index $d = 12g - 12 + 6p$. By restricting the representations constructed in Theorem 4.3 (ii) to this subgroup, we can construct a one parameter family complex hyperbolic quasi-Fuchsian representations of π_1 to $\mathrm{SU}(2,1)$. It remains to compute the Toledo invariants of these representations. Using the discussion above and Proposition 4.4 the Toledo invariant of the representation corresponding to the parameter $\mathbb{A}$ is $\tau = \mathbb{A}d/3\pi = (4g - 4 + 2p)\mathbb{A}/\pi$ where $\mathbb{A}$ varies between $-\pi/2$ and $\pi/2$. Hence τ varies between $-2g + 2 - p = \chi$ and $2g - 2 + p = -\chi$. $\square$

We remark that similar arguments show that the representations constructed in Theorem 4.3 (iii) also have Toledo invariant $\tau = \mathbb{A}/3\pi$. Therefore a similar argument may be used to construct representations of punctured surface groups that interpolate between $\mathbb{C}$-Fuchsian groups and groups with accidental parabolics. Moreover, if τ_1 is the Toledo invariant of these limit groups then $\tan(\pi\tau_1/2\chi) = \mp\sqrt{15}$.

4.3 A special case: the once punctured torus

In his thesis [67], Will considered the case where $g = p = 1$, that is the once punctured torus. In this section Σ will denote the once punctured torus and π_1 its fundamental group. Consider an $\mathbb{R}$-Fuchsian representation ρ_0 of π_1. This is generated by two loxodromic maps A and B whose axes intersect and whose commutator is parabolic. Suppose $\rho_0(\pi_1)$ fixes the Lagrangian plane R. Let R_0 be the Lagrangian plane orthogonal to R through the intersection of the axes of A and B. Let I_0 denote the (anti-holomorphic) involution fixing R_0. Then I_0 conjugates A to A^{-1} and B to B^{-1}. $I_1 = I_0A$ and $I_2 = BI_0$ are also antiholomorphic involutions. We see that $\rho_0(\pi_1) = \langle A, B\rangle$ is an index 2 subgroup of the group $\langle I_0, I_1, I_2\rangle$. If a representation ρ of π_1 is an index 2 subgroup of a group generated by three anti-holomorphic involutions, then we say ρ is *Lagrangian decomposable*. Will is able to construct a three dimensional family of Lagrangian decomposable punctured torus groups:

Theorem 4.7 (Will [67]) *Let* Σ *be the once punctured torus and let* $\mathcal{T}(\Sigma)$ *denote its Teichmüller space. Then for all* $x \in \mathcal{T}(\Sigma)$ *and all* $\alpha \in [-\pi/2, \pi/2]$ *there is a representation* $\rho_{x,\alpha} : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ *with the following properties:*

(i) *$\rho_{x,\alpha}$ and $\rho_{y,\beta}$ are conjugate if and only if $x = y$ and $\alpha = \beta$.*

(ii) *For all $x \in \mathcal{T}(\Sigma)$ and all $\alpha \in [-\pi/4, \pi/4]$ then $\rho_{x,\alpha}$ is complex hyperbolic quasi-Fuchsian.*

(iii) *$\rho_{x,\alpha}$ is $\mathbb{R}$-Fuchsian if and only if $\alpha = 0$.*

(iv) *If $\alpha \in [-\pi/2, -\pi/4)$ or $(\pi/4, \pi/2]$ then there exists an $x \in \mathcal{T}(\Sigma)$ so that $\rho(x, \alpha)$ is either not faithful or not discrete.*

It is easy to deduce from the details of Will's construction, that $\tau(\rho_{x,\alpha}) = 0$ for all $x \in \mathcal{T}(\Sigma)$ and all $\alpha \in [-\pi/2, \pi/2]$.

Sketch of proof. Will's basic idea is to take an ideal triangle in the hyperbolic plane and to consider points moving along each of the three boundary arcs. These points are parametrised by the signed distance t_0, t_1 and t_2 respectively from point that is the orthogonal projection of the opposite vertex onto this edge. These three points will be the fixed points of involutions I_0, I_1 and I_2. He shows that the product $(I_0I_1I_2)^2$ will be parabolic (fixing one of the vertices of the triangle) if and only if $t_1 + t_2 + t_3 = 0$. If this condition is satisfied then the group generated by $A = I_0I_1$ and $B = I_2I_0$ is a representation of a punctured torus group to and any two of the lengths parametrise $\mathcal{T}(\Sigma)$.

We embed the picture above in a Lagrangian plane R in $\mathbf{H}^2_{\mathbb{C}}$. We construct Lagrangian planes R_0, R_1 and R_2 through the three points on the sides of the triangle and which each make an angle $\pi/2 + \alpha$ with R, in a sense which he makes precise. The (anti-holomorphic) involution I_j fixes R_j for $j = 0$, 1, 2. This then gives a representation of π_1 to $\mathrm{SU}(2, 1)$ generated by $A = I_0I_1$ and $B = I_2I_0$. The restriction of the action of this group to R is just the action constructed in the previous paragraph and only depends on t_0, t_1 and α. This is the representation $\rho_{x,\alpha}$.

When $\alpha \in [-\pi/4, \pi/4]$ Will constructs a fundamental domain for the action of $\langle I_0, I_1, I_2\rangle$ on $\mathbf{H}^2_{\mathbb{C}}$. This domain is bounded by three packs each of whose slices makes an angle $\pi/2+\alpha$ with R. The intersection of this domain with R is just the triangle with which we began.

When $\alpha \in [-\pi/2, -\pi/4)$ or $\alpha \in (\pi/4, \pi/2]$ Will considers groups with $t_0 = t_1$ and he shows that for large $t_0 = t_1$ then $A = I_0I_1$ is elliptic. Indeed, writing $t = t_0 = t_1$ he shows that $\mathrm{tr}(A) = 3+e^{-4t}+4e^{-2t}\cos(2\alpha)$. For these values of α we have $\cos(2\alpha) < 0$ and so A is parabolic when $e^{-2t} = -\cos(2\alpha)$ and elliptic when $e^{-2t} < -\cos(2\alpha)$. $\square$

4.4 Disconnectedness of complex hyperbolic quasi-Fuchsian space

Dutenhefner and Gusevskii proved in [13] that in the non compact case, there exist complex hyperbolic quasi-Fuchsian representations of a the fundamental group of a surface Σ which cannot be connected to a Fuchsian representation by a path lying entirely in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$. Therefore, the complex hyperbolic quasi-Fuchsian space $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ is not connected.

Theorem 4.8 (Dutenhefner and Gusevskii [13]) *There exist complex hyperbolic quasi-Fuchsian representations of π whose limit set is a wild knot.*

Outline of proof. In the first place, the authors construct a complex hyperbolic quasi-Fuchsian group whose limit set is a wild knot. To do so, they consider a non trivial knot K (in fact a granny knot) inside the Heisenberg group $\mathfrak{N}$ and a finite collection $S_k, \dots, S'_k$, $k = 1, \dots, n$ of Heisenberg spheres placed along K such that there exists an enumeration $T_1, \dots, T_{2n}$ of the spheres of this family such that each T_k is tangent to T_{k-1} and T_{k+1} and is disjoint from all the rest (we take the indices cyclically). and T_{2n} and T_1 which are tangent. They call such a collection a *Heisenberg string of beads.* They prove then that there exist $A_k \in \mathrm{SU}(2,1)$ such that

- $A_k(S_k) = S'_k$, $k = 1, \dots, n$,
- A_k map the exterior of S_k into the interior of S'_k, $k = 1, \dots, n$ and
- A_k maps the points of tangency of S_k to the points of tangency of S'_k, $k = 1, \dots, n$

Consider the group $\Gamma = \langle A_1, \dots, A_n \rangle$. By Poincaré's polyhedron theorem, Γ is complex hyperbolic quasi-Fuchsian and its limit set is a wild knot. $\square$

5 Closed surfaces

We now turn our attention to the case of closed surfaces Σ without boundary, that is the case $p = 0$ and $g \geq 2$.

5.1 The representation variety

We begin by discussing the representation variety of π_1, the fundamental group of Σ. In particular, for the moment we do not consider discreteness. There are some differences between the case of representations to

SU(2, 1) and to PU(2, 1). That is to say, between the representation varieties $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ and $\mathrm{Hom}(\pi_1, \mathrm{PU}(2,1))/\mathrm{PU}(2,1)$. Mostly, we concentrate on the case of representations to SU(2, 1). Many of our results go through in both cases. For our first result, we distinguish between the two cases:

Proposition 5.1 (Goldman, Kapovich, Leeb [28]) *Let π_1 be the fundamental group of a closed surface of genus $g \geq 2$.*

(i) Let $\rho : \pi_1 \longrightarrow \mathrm{PU}(2,1)$ be a representation of π_1 to $\mathrm{PU}(2,1)$. Then $\tau \in \frac{2}{3}\mathbb{Z}$.

(i) Let $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ be a representation of π_1 to $\mathrm{SU}(2,1)$. Then $\tau \in 2\mathbb{Z}$.

Because τ is locally constant and varies continuously (with respect to the compact-open topology) with ρ then we immediately have:

Corollary 5.2 *The Toledo invariant is constant on components of* $\mathrm{Hom}(\pi_1, \mathrm{PU}(2,1))/\mathrm{PU}(2,1)$ *or* $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$. *That is, if ρ_1 and ρ_2 are representations with $\tau(\rho_1) \neq \tau(\rho_2)$ then ρ_1 and ρ_2 are in different components of the representation variety.*

Now, there is a converse to Corollary 5.2 due to Xia, see [71]. Namely, that the Toledo invariant τ distinguishes the components of the whole representation variety.

Theorem 5.3 (Xia [71]) *If $\tau(\rho_1) = \tau(\rho_2)$ then ρ_1 and ρ_2 lie in the same component of the representation variety* $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$.

We know that $\tau(\rho) \in [\chi, -\chi] = [2-2g, 2g-2]$ and (for the SU(2, 1) representation variety) $\tau(\rho) \in 2\mathbb{Z}$. Therefore $\tau(\rho)$ takes one of the $2g-1$ values $2-2g$, $4-2g$, $\ldots$, $2g-4$, $2g-2$. For each one of these values of $\tau(\rho)$ there is a component of $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$.

5.2 $\mathbb{C}$-Fuchsian representations

Let us consider the components for which $\tau(\rho) = \mp\chi = \pm(2g-2)$. Suppose that ρ_0 is a $\mathbb{C}$-Fuchsian representation. Then, using Proposition 3.8 we know that $\tau(\rho_0) = \mp\chi$. Because τ is constant on the components of $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$, we see that any deformation ρ_t of ρ_0 also has $\tau(\rho_t) = \tau(\rho_0) = \mp\chi$. Therefore, ρ_t is also $\mathbb{C}$-Fuchsian, again using Proposition 3.8. Thus we have proved the following result, known as the *Toledo-Goldman rigidity theorem*:

Theorem 5.4 (Goldman [25], Toledo [64, 65]) *Given a $\mathbb{C}$-Fuchsian representation $\rho_0 \in \mathcal{Q}_{\mathbb{C}}(\Sigma)$ then any nearby representation ρ_t is also $\mathbb{C}$-Fuchsian.*

In fact, we can give a complete description of this component. Let $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ be a $\mathbb{C}$-Fuchsian representation of π_1. This means that $\rho(\pi_1)$ preserves a complex line L. We may suppose that L is $\{0\} \times \mathbb{C} \subset \mathbb{C}^2$, that is the z_2 axis, in the unit ball model of $\mathbf{H}^2_{\mathbb{C}}$. This means that ρ is a reducible representation

$$\rho : \pi_1 \longrightarrow \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1,1)) < \mathrm{SU}(2,1).$$

That is, each element $A \in \rho(\pi_1)$ is a block diagonal matrix. The upper left block A_1 is a 1×1 block in $\mathrm{U}(1)$, in other words $A_1 = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. The lower right hand block is a 2×2 block A_2 in $\mathrm{U}(1,1)$ with determinant $e^{-i\theta}$. This means that $e^{i\theta/2}A_2 \in \mathrm{SU}(1,1)$. Hence we can write $\rho = \rho_1 \oplus \rho_2$ where $\rho_1 : \pi_1 \longrightarrow \mathrm{U}(1)$ is given by $\rho(\gamma) = A_1$ and $\rho_2 : \pi_1 \longrightarrow \mathrm{U}(1,1)$ is given by $\rho(\gamma) = A_2$. This is equivalent to $\rho_2' : \pi_1 \longrightarrow \mathrm{SU}(1,1)$ is given by $\rho(\gamma) = e^{i\theta/2}A_2$. Thus ρ_1 is an abelian representation and ρ_2 is (a lift of) a Fuchsian representation. Note that after applying the canonical projection from $\mathrm{U}(1,1)$ to $\mathrm{PU}(1,1)$ the resulting Möbius transformation is independent of $\det(A_2)$. Thus as Fuchsian representations ρ_2 and ρ_2' are the same. The representations ρ_1 and ρ_2' are independent. The space of (irreducible) Abelian representations is $2g$ copies of S^1, that is a $2g$-dimensional torus T^{2g} (note that since the only relation is a product of commutators, we have a free choice of points in S^1 for each of the $2g$ generators of π_1). The space of Fuchsian representations up to conjugacy is Teichmüller space $\mathcal{T}(\Sigma)$, which is homeomorphic to $\mathbb{R}^{6g-6}$. Thus we have

Proposition 5.5 (Goldman [26]) *Let Σ be a closed surface of genus $g \geq 2$. The two components of $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ where $\tau = \mp\chi = \pm(2g-2)$ are made up of discrete faithful Fuchsian representations in $T^{2g} \times \mathcal{T}(\Sigma)$. In particular, these two components have dimension $8g-6$.*

Every representation with $\tau \neq \mp\chi$ is irreducible. One may compute the dimension using Weyl's formula to obtain:

Proposition 5.6 (Goldman [26]) *Let Σ be a closed surface of genus $g \geq 2$. Each of the $2g-3$ components of $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ for which $\tau \neq \mp\chi = \pm(2g-2)$ has dimension $16g-16$.*

In contrast to the two components of $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ with $\tau = \mp\chi$, the other components of the representation variety are not easy to describe. In particular, they contain representations that are not quasi-Fuchsian. We expect the picture is that each of these other components contains (probably infinitely many) islands of quasi-Fuchsian representations; see Problem 6.1. As yet very little is known. In the following sections we will summarise the current state of knowledge.

5.3 Complex hyperbolic quasi-Fuchsian space is open

We use the following theorem of Guichard.

Theorem 5.7 (Guichard [31]) *Let G be a semi-simple group with finite centre and let G^* be a subgroup of G of rank 1. If Γ is a convex-cocompact subgroup of G^* then there exists an open neighbourhood U of an injection of Γ in G into the space of representations $\mathrm{Hom}(\Gamma, G)/G$ consisting of discrete, faithful, convex-cocompact representations.*

Guichard proves this result using the theory of Gromov hyperbolic metric spaces. In particular, he shows that there is a neighbourhood of Γ comprising quasi-isometric embeddings of Γ into G. Using a theorem of Bourdon and Gromov [7] he then deduces that the groups in this neighbourhood are discrete, faithful and convex-cocompact.

Corollary 5.8 (Guichard [31]) *Let Σ be a closed surface of genus g and let $\rho_0 : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ be a complex hyperbolic quasi-Fuchsian representation of π_1. Then there exists an open neighbourhood $U = U(\rho_0)$ so that any representation $\rho \in U$ is complex hyperbolic quasi-Fuchsian.*

Combining this Corollary 5.8 and Proposition 5.6 we obtain

Corollary 5.9 *There are open sets of dimension $16g - 16$ in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.*

In [48] Parker and Platis proved a version of Corollary 5.8 in the case where ρ_0 is $\mathbb{R}$-Fuchsian. They started with a fundamental domain for the group $\rho_0(\pi_1)$ whose faces are contained in packs. This domain is the preimage under orthogonal projection of a fundamental polygon for the action of $\rho_0(\pi_1)$ on its invariant Lagrangian plane. They then showed directly that this fundamental domain may be continuously deformed into a fundamental domain for $\rho(\pi_1)$. Such a domain has faces contained in packs and has the same combinatorial type as the original domain.

5.4 The Euler number

Let $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ be a complex hyperbolic quasi-Fuchsian representation of π_1. Let $M = \widetilde{f}(\widetilde{\Sigma})$ be an equivariant surface defined by $\widetilde{f} : \widetilde{\Sigma} \longrightarrow M \subset \mathbf{H}^2_{\mathbb{C}}$. The quotient of M by $\Gamma = \rho(\pi_1)$ is a surface homeomorphic to Σ and $\mathbf{H}^2_{\mathbb{C}}/\Gamma$ is a disc bundle over this surface. The relation in π_1 corresponds to moving around the boundary of a fundamental domain for the action of Γ on M. In doing so it is easy to check that a tangent vector rotates by a total angle of $2\pi\chi = (4-4g)\pi$, that is $\chi = 2-2g$ whole turns. As this happens the discs in the normal direction also rotate by a certain number of whole turns. This number is called the *Euler number* $e = e(\rho)$ of ρ. The Euler number measures how far $\mathbf{H}^2_{\mathbb{C}}/\Gamma$ is from being a product of Σ with a disc. We now demonstrate how to find the Euler number for $\mathbb{C}$-Fuchsian and $\mathbb{R}$-Fuchsian representations.

An elliptic map in $\mathrm{SU}(1,1)$ fixing the origin in the ball model lifts to $\mathrm{S}(\mathrm{U} \times \mathrm{U}(1,1))$ as

$$A_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta/2} & 0 \\ 0 & 0 & e^{-i\theta/2} \end{bmatrix}.$$

This acts on $(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}}$ via its standard lift as $A_\theta : (z_1, z_2) \longmapsto (e^{i\theta/2}z_1, e^{i\theta}z_2)$. In this case M is the (complex) z_2 axis. We may identify tangent vectors to M at the origin with vectors $(0, z_2)$ and normal vectors to M at the origin with vectors $(z_1, 0)$. Hence A_θ acts as rotation by θ on vectors tangent to M and by rotation through $\theta/2$ on vectors normal to M. Suppose $\rho(\pi_1) < \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1,1)) < \mathrm{SU}(2,1)$ is a $\mathbb{C}$-Fuchsian representation. The relation in π_1 corresponds to a rotation of a tangent vector to M by $\chi = 2-2g$ whole turns (that is by an angle of $(4-4g)\pi$). Hence the normal vector makes half this number of turns. Thus the Euler number is $e(\rho) = 1 - g = \chi/2$.

An elliptic map in $\mathrm{SO}(2,1)$ fixing the origin in the ball model embeds in $\mathrm{SO}(2,1)$ as

$$B_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This acts on $(z_1, z_2) \in \mathbf{H}^2_{\mathbb{C}}$ via its standard lift as $B_\theta : (z_1, z_2) \longmapsto (\cos\theta z_1 - \sin\theta z_2, \sin\theta z_1 + \cos\theta z_2)$. In this case M is the set where z_1 and z_2 are both real. We may identify tangent vectors to M at the origin with vectors (x_1, x_2) and normal vectors to M at the origin with vectors (iy_1, iy_2) where x_1, x_2, y_1, y_2 are all real. It is clear that tangent vectors and normal vectors are all rotated through angle θ. Suppose $\rho(\pi_1) < \mathrm{SO}(2,1) < \mathrm{SU}(2,1)$ is an $\mathbb{R}$-Fuchsian representation. Since the relation in π_1 corresponds to a rotation of of a tangent vector to M of $\chi = 2 - 2g$ whole turns. Hence the normal vector makes the same number of turns. Thus the Euler number is $e(\rho) = 2 - 2g = \chi$.

5.5 Construction of examples

Besides rigidity Theorem 5.4 and Proposition 5.5 which describe completely the components of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ comprising of $\mathbb{C}$-Fuchsian representations, none of the above results imply anything about discreteness. The first result towards this direction is due to Goldman, Kapovich and Leeb, see Theorem 1.2 of [28]:

Theorem 5.10 (Goldman, Kapovich and Leeb [28]) *Let Σ be a closed surface of genus g and let π_1 be its fundamental group. For each even integer t with $2 - 2g \leq t \leq 2g - 2$ there exists a convex-cocompact discrete and faithful representation ρ of π_1 with $\tau(\rho) = t$. Furthermore, the complex hyperbolic manifold $M = \mathbf{H}^2_{\mathbb{C}}/\rho(\pi_1)$ is diffeomorphic to the total space of an oriented $\mathbb{R}^2$ bundle ξ over Σ with the Euler number*

$$e(\xi) = \chi(\Sigma) + |\tau(\rho)/2|.$$

Outline of proof. We have already seen that $\mathbb{R}$-Fuchsian or $\mathbb{C}$-Fuchsian representations give examples of representations with $\tau = 0$ or $\tau = \pm(2g-2)$ and the correct Euler numbers.

Let t be an even integer with $0 < t < 2g - 2$. Let Σ_1 be a (possibly disconnected) subsurface of Σ with the following properties:

(i) $\chi(\Sigma_1) = -t$,

(ii) each boundary component is an essential simple closed curve in Σ and distinct boundary components are not homotopic,

(iii) each component of Σ_1 has an even number of boundary components.

It is easy to see that for each even integer t with $0 < t < 2g - 2$ we can can find such a Σ_1. Let Σ_2 be the interior of $\Sigma - \Sigma_1$. Then Σ_2 is also a (possibly disconnected) subsurface of Σ.

The authors start with a piecewise hyperbolic structure on Σ with the following properties. First, Σ_1 has constant curvature -1 and Σ_2 has constant curvature $-1/4$. Secondly, each boundary component of Σ_1 and Σ_2

is a geodesic with the same length l which is sufficiently short, they take $\sinh(l) < 6/35$. Next, the main step for the proof is the construction of a complete complex hyperbolic manifold M together with a piecewise totally geodesic isometric embedding $f : \Sigma \to M$ such that

(i) f is a homotopy equivalence,

(ii) the Toledo invariant of M equals $-\chi(\Sigma_1) = t$ and

(iii) M is diffeomorphic to the total space of an oriented disc bundle which has the Euler number $\chi(\Sigma_1)/2 + \chi(\Sigma_2)$.

The method they use to construct M is obtained by taking a $\mathbb{C}$-Fuchsian representation of the fundamental group of (each component of) Σ_1 and an $\mathbb{R}$-Fuchsian representation of the fundamental group of (each component of) Σ_2. They then use a version of the Klein-Maskit combination theorem to sew these representations together. The hypothesis that all the boundary components are short guarantees that the resulting representation is discrete.

Finally, for even integers t with $2 - 2g < t < 0$, the required representation is obtained by applying an antiholomorphic isometry to the representation constructed above with $\tau = -t$. □

The reason that Σ_1 is required to have an even number of components (and hence $\chi(\Sigma_1)$ is even) is rather subtle. In order to be able to use the Klein-Maskit combination theorem, each curve γ in the common boundary component of Σ_1 and Σ_2 must be represented by an element A of $\mathrm{SU}(2,1)$ that is (up to conjugation) simultaneously a loxodromic element of $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1,1))$ and of $\mathrm{SO}(2,1)$. In particular it has real trace greater than $+3$. Hence, the $\mathrm{U}(1)$ part of its representation in $\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1,1))$ must be $+1$ and hence it lies in $\{+1\} \times \mathrm{SU}(1,1)$ and corresponds to an element of $\mathrm{SU}(1,1)$ with trace greater than $+2$.

Suppose that Σ_0 is a three-holed sphere with boundary components γ_1, γ_2 and γ_3. For any choice of hyperbolic metric on Σ_0 we may associate a geometrical representation $\rho_0 : \pi_1(\Sigma_0) \longrightarrow \mathrm{PU}(1,1)$. Consider any lift of ρ_0 to $\mathrm{SU}(1,1)$. Let A_1, A_2 and A_3 be the three elements of $\mathrm{SU}(1,1)$ representing the boundary loops. Then either all three traces are negative or else one is negative and the other two positive (see page 9 of Gilman [23] for example). This means that for at least one of the boundary components the corresponding element of $\mathrm{SU}(2,1)$ has trace less than -1, and hence it corresponds to a glide reflection in $\mathrm{SO}(2,1)$. By studying the decomposition of Σ_1 into three holed spheres, we can see that each boundary component may be represented by an element of $\mathrm{SU}(1,1)$ positive trace if and only if the number of boundary components is even.

Further examples have been constructed by Anan'in, Grossi and Gusevskii [2, 3]. In their construction, they consider a group generated by complex involutions $R_1, \ldots, R_n$ fixing complex lines $L_1, \ldots, L_n$ respectively with two properties. First, the complex lines are pairwise ultra-parallel and, secondly, the product $R_1 \ldots R_n$ is the identity. They give conditions that determine whether such a group is discrete and they do so by constructing a fundamental polyhedron whose faces are contained in bisectors. These groups contain a complex hyperbolic quasi-Fuchsian subgroup of index 2 or 4. In particular they give a series of examples where the Euler number takes different values. Subsequently, a related construction was given by Gaye [22]. He considers groups of the same type as those considered in [2] but he constructs fundamental polyhedra whose faces are contained in $\mathbb{C}$-spheres.

5.6 Complex hyperbolic Fenchel-Nielsen coordinates

It is useful to find ways of putting coordinates in complex hyperbolic quasi-Fuchsian space. One way to do it is to mimic the construction of Fenchel-Nielsen coordinates for Teichmüller space and the related complex Fenchel-Nielsen coordinates for quasi-Fuchsian space.

For clarity, we shall recall in brief how Fenchel-Nielsen coordinates are defined; see [21] or Wolpert [68], [69]. Let γ_j for $j = 1, \ldots, 3g-3$ be a *curve system* (some authors call this a *partition*) that is, a maximal collection of disjoint, simple, closed curves on Σ that are neither homotopic to each other nor homotopically trivial. The complement of such a curve system is a collection of $2g-2$ three-holed spheres. If Σ has a hyperbolic metric then, without loss of generality, we may choose each γ_j in our curve system to be the geodesic in its homotopy class. The hyperbolic metric on each three-holed sphere is completely determined by the hyperbolic length $l_j > 0$ of each of its boundary geodesics. There is a real twist parameter k_j that determines how these three-holed spheres are attached to one another. From an initial configuration, the two three holed spheres with common boundary component may be rotated relative to one another. The absolute value of the twist $|k_j|$ measures the distance they are twisted and the sign denotes the direction of their relative twist. The theorem of Fenchel and Nielsen states that each $(6g-6)$-tuple

$$(l_1, \ldots, l_{3g-3}, k_1, \ldots, k_{3g-3}) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$$

determines a unique hyperbolic metric on Σ and each hyperbolic metric arises in this way.

Kourouniotis [39] and Tan [63] extended the theorem of Fenchel and Nielsen to the case of real hyperbolic quasi-Fuchsian space of Σ. In this extension both the length parameters and the twist parameters become complex. The elements of $\rho(\pi_1) \in \mathrm{SL}(2, \mathbb{C})$ corresponding to γ_j in the curve system are now loxodromic with, in general, non-real trace. Thus the imaginary part of the length parameter represents the holonomy angle when moving around γ_j. Likewise, the imaginary part of the twist parameter becomes the parameter of a bending deformation about γ_j; see also [51] for more details of this correspondence and how to relate these parameters to traces of matrices. The main difference from the situation with real Fenchel-Nielsen coordinates is that, while distinct quasi-Fuchsian representations determine distinct complex Fenchel-Nielsen coordinates, it is not at all clear which set of coordinates give rise to discrete representations, and hence to a quasi-Fuchsian structure. In fact the boundary of the set of realisable coordinates is fractal.

In [49], Parker and Platis define Fenchel-Nielsen coordinates for representations $\pi_1(\Sigma)$ to $\mathrm{SU}(2,1)$ for which the $3g-3$ curves in a curve system are represented by loxodromic maps. It is clear that this is a proper subset of the representation variety and contains complex hyperbolic quasi-Fuchsian space. The coordinates are $16g - 16$ real parameters that distinguish non-conjugate irreducible representations and $8g-6$ real parameters that distinguish non-conjugate representations that preserve a complex line (compare to Propositions 5.5 and 5.6). As with the complex Fenchel-Nielsen coordinates described by Kourouniotis and Tan it is not clear which coordinates correspond to discrete representations. However, the coordinates in [49] determine the group up to conjugacy and distinguish between non-conjugate representations. The major innovation of Parker and Platis in this paper is the use of cross-ratios (recall section 2.7) in addition to complex length and twist-bend parameters. First, for representations that do not preserve a complex line, and so are irreducible, the following holds.

Theorem 5.11 (Parker and Platis [49]) *Let Σ be a surface of genus g with a simple curve system $\gamma_1, \ldots, \gamma_{3g-3}$. Let $\rho : \pi_1(\Sigma) \longrightarrow \Gamma < \mathrm{SU}(2,1)$ be an irreducible representation of the fundamental group $\pi_1(\Sigma)$ to $\mathrm{SU}(2,1)$ with the property that $\rho(\gamma_j) = A_j$ is loxodromic for each $j = 1, \ldots, 3g-3$. Then there exist $4g-4$ complex parameters and $2g-2$ points on the cross-ratio variety $\mathfrak{X}$ that completely determine ρ up to conjugation.*

For representations that preserve a complex line several of the previous parameters are real the following holds.

Theorem 5.12 (Parker and Platis [49]) *Let Σ be a surface of genus g with a simple curve system $\gamma_1, \ldots, \gamma_{3g-3}$. Let $\rho : \pi_1(\Sigma) \longrightarrow \Gamma < \mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1,1))$ be a reducible representation of the fundamental group $\pi_1(\Sigma)$ to $\mathrm{SU}(2,1)$ that preserves a complex line and which has the property that $\rho(\gamma_j) = A_j$ is loxodromic for each $j = 1, \ldots, 3g-3$. Then there exist $8g-6$ real parameters that completely determine ρ up to conjugation.*

5.7 The boundary of complex hyperbolic quasi-Fuchsian space

We now investigate what happens as we approach the boundary of quasi-Fuchsian space. Cooper, Long and Thistlethwaite [11] have proved a complex hyperbolic version of Chuckrow's theorem. They prove it for sequences of representations of groups that are not virtually nilpotent. We state it for the special case we are interested in here, namely the case of surface groups.

Theorem 5.13 (Theorem 2.7 of [11]) *Let π_1 be the fundamental group of a hyperbolic surface of finite type. Suppose that $\rho_k : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ is an algebraically convergent sequence of discrete, faithful representations of π_1. Then the limit representation is discrete and faithful.*

In Section 2.7 of [30] Greenberg used the classical version of Chuckrow's theorem to construct some of the first examples of geometrically finite Kleinian groups. Roughly speaking, his argument is the following: Consider a space X of discrete, faithful, geometrically finite, type-preserving $\mathrm{SL}(2,\mathbb{C})$-representations of a given abstract group so that X is open and the boundary of X in the space of all representations contains a continuum. The example Greenberg considers is Riley's example of groups generated by two parabolic maps, now known as the Riley slice of Schottky space. Then Greenberg considers a sequence of representations ρ_k converging to the boundary of X. Because X is open the limit representation ρ_0 is not in X. By Chuckrow's theorem ρ_0 is both discrete and faithful. Therefore ρ_0 must either contain additional parabolics or be geometrically infinite. This reasoning applies in our case. Greenberg goes on to argue that, since for a matrix to be parabolic in $\mathrm{SL}(2,\mathbb{C})$ involves a codimension 2 condition (trace equals ± 2), there are only countably many boundary points with additional parabolic elements. Therefore there must be geometrically infinite examples on the boundary. In the case of $\mathrm{SU}(2,1)$ the condition to be parabolic has codimension 1 ($f(\tau) = 0$ where $f(\tau)$ is Goldman's function Theorem 6.2.4 of [27]).

Therefore it could be the case that the boundary of complex hyperbolic quasi-Fuchsian space is made up of geometrically finite representations where (at least) one conjugacy class in π_1 is represented by parabolic elements of $\mathrm{SU}(2,1)$. The locus where a particular conjugacy class is parabolic is a real analytic codimension 1 subvariety (with respect to suitable coordinates, such as the complex hyperbolic Fenchel-Nielsen coordinates described in Section 5.6). Therefore a conjectural picture is that complex hyperbolic quasi-Fuchsian space has a polyhedral or cellular structure. The (real) codimension 1 cells correspond to a single extra class of parabolic maps. These intersect in lower dimensional cells. The cells of codimension k correspond to k classes becoming parabolic.

5.8 Complex hyperbolic quakebending

Let ρ_0 be a representation of the fundamental group π_1 of Σ into some Lie group G. A deformation of ρ_0 is a curve $\rho_t = \rho(t)$ such that $\rho(0) = \rho_0$. Deformations in Teichmüller and real quasi-Fuchsian spaces are very well known and have been studied extensively, at least throughout the last thirty years. Below we state some basic facts about them.

In the case of Teichmüller space $\mathcal{T}(\Sigma)$, the basic deformation is the Fenchel-Nielsen deformation; a thorough study of this has been carried out by Wolpert in [69]. We cut Σ_0 along a simple closed geodesic α, rotate one side of the cut relative to the other and attach the sides in their new position. This deformation involves continuously changing the corresponding Fenchel-Nielsen twist parameter while holding all the others fixed. The hyperbolic metric in the complement of the cut extends to a hyperbolic metric in the new surface. In this way a deformation ρ_t (depending on the free homotopy class of α) is defined and its infinitesimal generator t_α is the Fenchel-Nielsen vector field. Such vector fields are very important: at each point of $\mathcal{T}(\Sigma)$, $6g-6$ of such fields form a basis of the tangent space of $\mathcal{T}(\Sigma)$. Moreover, the Weil-Petersson Kähler form of $\mathcal{T}(\Sigma)$ may be described completely in terms of the variations of geodesic length of simple closed geodesics under the action of these fields. The basic formula for this is Wolpert's first derivative formula: If α, β are simple closed geodesics in Σ_0, l_α is the geodesic length of α and t_β is the Fenchel-Nielsen vector field associated to β then then at the point ρ_0 we have

$$t_\beta l_\alpha = \sum_{p\in\alpha\cap\beta} \cos(\phi_p),$$

where ϕ_p is the oriented angle of intersection of α and β at p. Another basic formula concerns the mixed variations $t_\beta t_\gamma l_\alpha$; the reader should see for instance [70] or [69] for details.

In [41] Kourouniotis, using ideas of Thurston and working in the spirit of Wolpert's construction of the Fenchel-Nielsen deformation, constructs a quasiconformal homeomorphism of the complex plane which he calls the *bending homeomorphism*. Given a hyperbolic structure on Σ, from this homeomorphism he obtains a quasi-Fuchsian structure on Σ.

Epstein and Marden took a different and much more general point of view in [15]. Given a hyperbolic structure ρ_0 on a closed surface Σ, then for every finite geodesic lamination Λ in Σ with complex transverse measure μ and a simple closed geodesic α in Σ_0, there exists an isometric map h, depending on α, of Σ_0 into a hyperbolic 3-manifold M_h (the *quakebend map*). The image of this map is a *pleated surface* Σ_h, that is a complete hyperbolic surface which may be viewed as the original surface bent along the leaves of the lamination in angles depending on the imaginary part of μ, with its flat pieces translated relative to the leaves in distances depending on the real part of μ. The pleated surface Σ_h is then the boundary of the convex hull of M_h. For small $t \in \mathbb{C}$, quakebending along Λ with transverse measure $t\mu$ produces injective homomorphisms of $\pi_1(\Sigma)$ into $\mathrm{PSL}(2,\mathbb{C})$ with quasi-Fuchsian image and in this way we obtain a deformation $\rho_{t\mu}$ (*the quakebend curve*) of quasi-Fuchsian space $\mathcal{Q}_{\mathbb{R}}(\Sigma)$ with initial point our given hyperbolic structure, that is a point in the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ. It is evident that the Fenchel-Nielsen deformation as well as Kourouniotis' bending deformation are special cases of the above construction; the first is induced from the case where μ is real (*pure earthquake*) and the second from the case where μ is imaginary (*pure bending*). Infinitesimal generators of quakebend curves are the holomorphic vector fields T_μ. If α is a simple closed geodesic in Σ_0 and in the case where λ is finite with leaves $\gamma_1, \dots, \gamma_n$, then at the point ρ_0 we have

$$T_\mu l_\alpha = \frac{dl(\rho_{t\mu})}{dt}(0) = \sum_{k=1}^{n} \Re(\zeta_k) \cdot \cos(\phi_k)$$

where $\zeta_k = \mu(\alpha \cap \gamma_k)$ and ϕ_k are the oriented angles of intersection of α and γ_k. This formula is a generalisation of Kerckhoff's formula when $\mu \in \mathbb{R}$, see [38]. Epstein and Marden also give formulae for the second derivative as well as generalisation of these in the case where Λ is infinite.

In [40] Kourouniotis revisits the idea of bending. Based on [15] and using his bending homeomorphism as in [41], he constructs quakebending curves in $\mathcal{Q}(\Sigma)$ but there, the initially point ρ_0 is a quasi-Fuchsian structure on Σ. Moreover in [42] he goes on to define the variations of the *complex length* λ_α of a simple closed curve under bending along (Λ, μ). Platis used Kourouniotis' results to describe completely the complex symplectic form of $\mathcal{Q}_{\mathbb{R}}(\Sigma)$ in [52, 53]. This form may be thought as the complexification of the Weil-Petersson symplectic form of $\mathcal{T}(\Sigma)$. We remark finally that generalisations of the derivative formulae were given for instance by Series in [62].

In [4], Apanasov took the point view of Kourouniotis in [41] to construct bending curves in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$. In fact he proved the following

Theorem 5.14 (Apanasov [4]) *Let ρ_0 be an $\mathbb{R}-$Fuchsian representation of π_1 and write $\Gamma_0 = \rho_0(\pi_1)$. Then for any simple closed geodesic $\alpha \in \mathbf{H}^2_{\mathbb{R}}/\Gamma_0$ and for sufficiently small $t \in \mathbb{R}$ there exists a (continuous) bending deformation ρ_t of ρ_0 induced by Γ_0-equivariant quasiconformal homeomorphisms F_t of $\mathbf{H}^2_{\mathbb{C}}$.*

Platis in [54] followed the strategy suggested by Epstein and Marden in [15]. Let ρ_0 be an $\mathbb{R}-$Fuchsian representation of π_1 with $\Gamma_0 = \rho_0(\pi_1)$. Then, $M_0 = \mathbf{H}^2_{\mathbb{C}}/\Gamma_0$ is a complex hyperbolic manifold and embedded in M_0 there is a hyperbolic surface $\Sigma_0 = \mathbf{H}^2_{\mathbb{R}}/\Gamma_0$. For every finite geodesic lamination Λ in Σ with complex transverse measure μ and a simple closed geodesic α in Σ_0, there is an isometric map $B_{\mathbb{C}}$, depending on α, of M_0 into a complex hyperbolic manifold M_h. (the *complex hyperbolic quakebend map*). Restricted to Σ_0 the image of this map is a pleated surface Σ_h, something which is entirely analogous to the classical case. The pleated surface Σ_h is naturally embedded in M_h. By Corollary 5.8, for small $t \in \mathbb{R}$, complex hyperbolic quakebending along Λ with transverse measure $t\mu$ produces complex hyperbolic quasi-Fuchsian groups. That is,

Theorem 5.15 (Platis [54]) *There is an $\epsilon > 0$ such that for all t with $|t| < \epsilon$ the complex hyperbolic quakebend curve $\rho_{t\mu}$ lies entirely in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.*

Platis also discusses the variations of the complex hyperbolic length $\lambda = l + i\theta$ of $\rho_{t\mu}$. The induced formulae are natural generalisations of Epstein-Marden's formulae. For instance, for the first derivative of λ at ρ_0 we have the following.

$$\frac{dl}{dt}(0) = \sum_{k=1}^{n} \Re(\zeta_k) \cdot \cos(\phi_k),$$

$$\frac{d\theta}{dt}(0) = \sum_{k=1}^{n} \Im(\zeta_k) \cdot \frac{3\cos^2(\phi_k) - 1}{2}.$$

Remark 5.16 *We remark that in view of Corollary 5.8 and Theorem 5.10, the condition ρ_0 is $\mathbb{R}-$Fuchsian is now quite restrictive. Complex hyperbolic quakebending curves may be constructed in exactly the same way at least when the starting point ρ_0 is an arbitrary point of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ with even Toledo invariant $\tau(\rho_0)$. This construction is carried out in [55].*

6 Open problems and conjectures

Below we list a number of open problems concerning both compact and non compact cases. We also state a number of conjectures.

Problem 6.1 *Describe the topology of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$. For example:*

(i) In the case when $p = 0$ describe the components of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ with the same Toledo invariant.

(ii) For $|\tau(\rho)| < 2g - 2$, is each of these components homeomorphic to a ball of dimension $16g - 16$?

(iii) In the case when $p > 0$ describe the components of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.

(iv) For $p > 0$ is $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ an open subset of $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$?

(v) Is each component homeomorphic to a ball?

We have seen that $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ is disconnected. In the compact case the Toledo invariant distinguishes components of the representation variety $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ [71] and each component is non-empty [28]. The Euler number gives a finer classification of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$. As yet it s not known which values of the Euler number can arise; compare [2]. When $p > 0$ there are quasi-Fuchsian representations whose limit set is a wild knot [13]. It is natural to ask which wild knots arise in this way and whether this construction extends to the case of $p = 0$.

Problem 6.2 *Describe representations in $\partial\mathcal{Q}_{\mathbb{C}}(\Sigma)$. For example:*

(i) Does $\partial\mathcal{Q}_{\mathbb{C}}(\Sigma)$ admit a cell structure where each cell of codimension k corresponds to k elements of π_1 being represented by parabolic maps?

(ii) If the answer to (i) is positive then characterise which elements of π_1 correspond to codimension 1 cells and which combinations correspond to cells of higher codimension?

(iii) Is every representation in $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ geometrically finite?

(iv) Does every quasi-Fuchsian representation of a punctured surface arise in the boundary of the quasi-Fuchsian space of a closed surface?

It is tempting to suggest that the codimension 1 cells correspond exactly to the simple closed curves on Σ. However, the case of the three times punctured sphere, Theorem 4.1, suggests that there are other curves that may be pinched in this way.

A starting point could be to consider surface subgroups of complex hyperbolic triangle groups. Let q_1, q_j, q_3 integers at least 2 (possibly including ∞) and consider three involutions I_j each fixing a complex line so that $I_{j+1}I_{j-1}$ is elliptic of order q_j when q_j is finite and parabolic when q_j is ∞ (all indices are taken mod 3). Then $\langle I_1, I_2, I_3\rangle$ is a representation of a (q_1, q_2, q_3) triangle reflection group.

Conjecture 6.3 (Schwartz [59]) *Let q_1, q_2 and q_3 be integers at least 2 and let $\langle I_1, I_2, I_3\rangle < \mathrm{SU}(2,1)$ be the corresponding representation of the (q_1, q_2, q_3) triangle reflection group. Then $\langle I_1, I_2, I_3\rangle$ is a discrete, faithful, type-preserving, geometrically finite representation if and only if $I_1I_2I_3$ and $I_jI_{j+1}I_jI_{j-1}$ for $j = 1, 2, 3$ are all loxodromic.*

Moreover, the values of the q_j determine which of these four words becomes parabolic first.

The Coxeter group generated by reflections in the sides of a hyperbolic triangle contains a surface group as a torsion-free, finite index subgroup (indeed it contains infinitely many such surface groups). Therefore every discrete, faithful, type-preserving, geometrically finite representation $\langle I_1, I_2, I_2\rangle$ contains quasi-Fuchsian subgroups of finite index. If Schwartz's conjecture is true then the subgroups of the corresponding representations where one of $I_1I_2I_3$, $I_jI_{j+1}I_jI_{j-1}$ is parabolic will lie in $\partial\mathcal{Q}_{\mathbb{C}}(\Sigma)$.

There are also natural questions about the geometric and analytical structures of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$. Goldman [24] and Hitchin [34] have shown that the representation variety $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ admits natural symplectic and complex structures. Our next problem concerns these structures.

Problem 6.4 *Let Σ be a closed surface. Examine geometrical and analytical structures of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.*

(i) *Do the natural symplectic and complex structures on* $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/\mathrm{SU}(2,1)$ *given by Goldman and Hitchin naturally pass to* $\mathcal{Q}_{\mathbb{C}}(\Sigma)$*?*

(ii) *Take a complex hyperbolic quakebending deformation with starting point any irreducible representation. From this, define then geometrically a symplectic structure in the whole* $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ *such that it agrees with the Weil-Petersson symplectic form of Teichmüller space at* $\mathbb{R}$*-Fuchsian representations.*

(iii) *What can you say about the complex structure? For instance do complex hyperbolic Fenchel-Nielsen coordinates induce such a complex structure on* $\mathcal{Q}_{\mathbb{C}}(\Sigma)$*?*

The answer to Problem 6.4 (i) is affirmative in the classical cases of $\mathrm{Hom}(\pi_1, \mathrm{SL}(2,\mathbb{R}))/SL(2,\mathbb{R})$ and $\mathrm{Hom}(\pi_1, \mathrm{SL}(2,\mathbb{C}))/SL(2,\mathbb{C})$. That is, complex and symplectic structures pass naturally to $\mathcal{T}(\Sigma)$ and $\mathcal{Q}_{\mathbb{R}}(\Sigma)$ respectively. We conjecture that the answer is also affirmative in the complex hyperbolic case. We also conjecture that the answer to Problem 6.4 (iii) is negative.

Conjecture 6.5 *For* $p = 0$ *there is a hyperkähler structure in* $\mathcal{Q}_{\mathbb{C}}(\Sigma)$*.*

Quasi-conformal maps are the major tool used to define Teichmüller space. These have been generalised to complex hyperbolic space by Mostow, Chapter 21 of [45]. Also, Korányi and Reimann [36] have developed an extensive theory of quasiconformal mappings on the Heisenberg group. These may be extended to $\mathbf{H}^2_{\mathbb{C}}$ [37]. However, it is not known whether these quasiconformal mappings are strong enough to describe the whole of $\mathcal{Q}_{\mathbb{C}}(\Sigma)$.

Conjecture 6.6 *For* $p = 0$ *any two representations in the same component of* $\mathcal{Q}_{\mathbb{C}}(\Sigma)$ *are quasiconformally conjugate.*

In the non compact case, by a theorem of Miner, see [44], type preserving $\mathbb{C}-$Fuchsian and $\mathbb{R}-$Fuchsian representations are never quasiconformally conjugate. In the compact case, the authors believe that there is strong evidence that this conjecture is true. For instance Aebischer and Miner showed in [1] that the complex quasi-Fuchsian space of a classical Schottky group has this property.

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