Minimal mutation-infinite quivers

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Workshop on Cluster Algebras
and finite dimensional algebras
Introduction

Mutations on quivers studied following the introduction of cluster algebras by Fomin and Zelevinsky in 2002.

This work follows:

★ Classification of minimal infinite-type diagrams by Seven published in 2007
★ Classification of mutation-finite quivers by Felikson, Shapiro and Tumarkin published in 2012
Quivers

directed (multi-)graphs with no loops or 2-cycles
Adjacency matrix
\[ A = (a_{i,j}) \text{ where } a_{i,j} = \#(i \rightarrow j) - \#(j \rightarrow i) \]
Mutations

Mutation is a function on the quiver which acts at a vertex $k$ through 3 steps:

1. For each pair of arrows $i \to k \to j$ add an arrow $i \to j$.
2. Reverse direction of arrows adjacent to $k$.
3. Remove any 2-cycles created in step (1).
Mutation examples
mutate at top vertex
Mutations are involutions.
Mutation at vertex $k$ takes an adjacency matrix $B = (b_{i,j})$ to $B' = (b'_{i,j})$ where

$$b'_{i,j} = \begin{cases} 
-b_{i,j} & \text{if } i = k \text{ or } j = k \\
 b_{i,j} + \frac{|b_{i,k}b_{k,j} + b_{i,k}b_{k,j}|}{2} & \text{otherwise}
\end{cases}$$
Mutation-equivalent
if there is a sequence of mutations
Mutation-finite
or conversely mutation-infinite
Partial ordering
on mutation-infinite quivers given by inclusion
Minimal mutation-infinite quivers
Mutations do not preserve
minimal mutation-infinite property
Any quiver containing a minimal mutation-infinite subquiver is necessarily mutation-infinite.

A complete classification would give a systematic approach to check whether any given quiver is mutation-infinite or not.
Ahmet Seven’s classification
of minimal infinite-type diagrams

Seven classified all the infinite-type diagrams such that removing a vertex yielded a finite-type diagram.
In their paper on classifying mutation-finite quivers, FST proved that there were no minimal mutation-infinite quivers with more than 10 vertices.
Distinguished family of minimal mutation-infinite quivers which are orientations of simply-laced Coxeter diagrams of hyperbolic Coxeter simplices.
Coxeter simplex
convex hull of $n + 1$ points

Considered inside spherical, Euclidean or hyperbolic space.

$n + 1$ hyper-planes $H_i$ with dihedral angles $\frac{\pi}{k_{ij}}$ (or possibly 0) between $H_i$ and $H_j$. 
Coxeter diagram
from simplex bounded by $H_i$ with angles $\frac{\pi}{k_{ij}}$

* vertex $i$ for each $H_i$
* edge $i - j$ with no weight when $k_{ij} = 3$
* edge $i - j$ with weight $k_{ij}$ when $k_{ij} > 3$
A Coxeter group can be constructed from a Coxeter diagram through the following presentation

\[ \langle s_i \mid s_i^2 = 1 = (s_i s_j)^{k_{ij}} \rangle. \]
Simply-laced Coxeter diagram
only have $k_{ij} = 2$ or 3

Coxeter diagram with no weighted edges.

Choosing an orientation of the edges gives a quiver.
Simply-laced Spherical Coxeter diagrams are Dynkin diagrams of type $A, D$ or $E$. 

- $A_n$: \[ \bullet \quad \bullet \quad \cdots \quad \bullet \quad \quad \underset{n}{\bullet} \quad \bullet \ \bullet \]
- $D_n$: \[ \bullet \quad \bullet \quad \cdots \quad \bullet \quad \underset{n-2}{\bullet} \quad \bullet \]
- $E_6$: \[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]
- $E_7$: \[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]
- $E_8$: \[ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \]

$n$, $n - 2$.
Simply-laced Euclidean Coxeter diagrams are affine Dynkin diagrams of type $\tilde{A}$, $\tilde{D}$ or $\tilde{E}$.

$\tilde{A}_n$: \[
\begin{array}{c}
\dot{\bullet} \\
\bullet \quad \bullet \quad \ldots \quad \bullet \\
\bullet
\end{array}
\]

$\tilde{D}_n$: \[
\begin{array}{c}
\dot{\bullet} \\
\bullet \quad \bullet \quad \ldots \quad \bullet \\
\bullet \quad \bullet \quad \ldots \quad \bullet
\end{array}
\]

$\tilde{E}_6$: \[
\begin{array}{c}
\bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet
\end{array}
\]

$\tilde{E}_7$: \[
\begin{array}{c}
\bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\]

$\tilde{E}_8$: \[
\begin{array}{c}
\bullet \\
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\]
Simply-laced Hyperbolic Coxeter simplices give diagrams satisfying:

* any subdiagram is either a Dynkin diagram or an affine Dynkin diagram, but the diagram itself is not.
Felikson, Shapiro and Tumarkin classified all mutation-finite quivers - (almost all) orientations of simply-laced Hyperbolic Coxeter diagrams do not lie in this classification.
Mutation-finite orientations
of hyperbolic Coxeter diagrams
Orientations of simply-laced Hyperbolic Coxeter diagrams are \textit{minimal} mutation-infinite quivers.

Orientations of Dynkin diagrams and affine Dynkin diagrams are mutation-finite.

Orientations of simply-laced Hyperbolic Coxeter diagrams are mutation-infinite.
Are these all?
Are these all?

For minimal mutation-infinite quivers with 4 and 5 vertices:

Yes
Patterns among the quivers
Moves
replace a subquiver while staying minimal mutation-infinite
Another example
and many more
A sink-source mutation does not affect the underlying unoriented graph of a quiver and does not change the mutation class of any subquivers.
Any minimal mutation-infinite quiver can be transformed through sink source mutations and at most 10 moves to either

★ a hyperbolic Coxeter simplex diagram
★ a double arrow quiver
★ an exceptional quiver
Any minimal mutation-infinite quiver can be transformed through sink-source mutations and at most 5 moves to either

★ a hyperbolic Coxeter simplex diagram
★ a double arrow quiver
★ an exceptional quiver
Double arrow quivers
Exceptional cases
Hyperbolic Coxeter diagrams
Any minimal mutation-infinite quiver can be transformed through sink source mutations and at most 10 moves to either

★ a hyperbolic Coxeter simplex diagram
★ a double arrow quiver
★ an exceptional quiver