THE CATEGORY OF FINITELY PRESENTED SMOOTH MOD p REPRESENTATIONS OF $GL_2(F)$.

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Abstract. Let F be a finite extension of $\mathbb{Q}_p$. We prove that the category of finitely presented smooth Z-finite representations of $GL_2(F)$ over a finite extension of $\mathbb{F}_p$ is an abelian subcategory of the category of all smooth representations. The proof uses amalgamated products of completed group rings.

1. Introduction

Let $\mathbb{F}$ be a finite field of characteristic p. If G is a locally profinite topological group, let $\mathcal{C}_{\mathbb{F}}(G)$ be the category of smooth representations of G over $\mathbb{F}$. Throughout this paper, if K is an open subgroup of such a group G then ind_K^G denotes induction with compact support modulo K.

Definition 1.1. Let V be a smooth $\mathbb{F}$-representation of a locally profinite group G. Then V is:

(1) **finitely generated** if for some compact open subgroup K of G there is a surjection of $\mathbb{F}[G]$-modules
$$\operatorname{ind}_K^G W \twoheadrightarrow V$$
for a smooth finite-dimensional $\mathbb{F}$-representation W of K;

(2) **finitely presented** if for some compact open subgroups K_1, K_2 of G there is an exact sequence
$$\operatorname{ind}_{K_1}^G W_1 \to \operatorname{ind}_{K_2}^G W_2 \to V \to 0$$
for W_1 and W_2 smooth finite-dimensional $\mathbb{F}$-representations of K_1 and K_2 respectively.

Let F be a finite extension of $\mathbb{Q}_p$. The purpose of this article is to prove:

Theorem 1.2. *The category of finitely presented smooth $\mathbb{F}$-representations of $SL_2(F)$ is an abelian subcategory of $\mathcal{C}_{\mathbb{F}}(SL_2(F))$.*

The same holds for the category of finitely presented smooth Z-finite representations of $GL_2(F)$.

This is Theorem 5.1 and Corollary 5.2 below. In fact, we prove the same result with F replaced by any finite dimensional division algebra over $\mathbb{Q}_p$.

The theorem is equivalent to the statement that the kernel[1] of any map between finitely presented smooth representations is itself finitely presented. If $\mathcal{C}_{\mathbb{F}}(SL_2(F))$ were the category of modules over a ring R, this would be the statement that

[1] and the cokernel, but this is automatic

R is a coherent ring. Indeed, we will prove the theorem by considering smooth $\mathbb{F}$-representations as modules over the amalgamated product

$$\mathbb{F}[[K]] *_{\mathbb{F}[[I]]} \mathbb{F}[[K']],$$

where $K = SL_2(\mathcal{O}_F)$, $K' = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix}$ for π a uniformising element of D, and $I = K \cap K'$. Then a result of Åberg [Å82] shows that, under certain conditions, an amalgamated product of coherent rings over a noetherian ring is itself coherent. *Throughout, unless otherwise stated, by 'module', 'noetherian' or 'coherent' we mean 'left module', 'left noetherian' or 'left coherent'.*

Finitely presented representations of $GL_2(F)$ were previous studied by Hu [Hu12], Vigneras [Vig11], and Schraen [Sch15].[2] In particular, [Vig11] Theorem 6 shows that a smooth *admissible* finitely presented representation of $GL_2(F)$ has finite length, and that all of its subquotients are also admissible and finitely presented. On the other hand, the main result of [Sch15] says that, if F is a quadratic extension of $\mathbb{Q}_p$, then an irreducible supersingular representation of $GL_2(F)$ admitting a central character is never finitely presented.

We are motivated by the construction (see [CEG$^+$16]) of a 'patched module' M_∞ that has an action of $G = GL_n(F)$ and, hopefully, interpolates the hypothetical p-adic Langlands correspondence. It is (only?) possible to directly obtain information about M_∞ by considering $\mathrm{Hom}_{GL_n(F)}(\mathrm{ind}_K^G(W), M_\infty^\vee)$ for locally algebraic representations of $K = GL_n(\mathcal{O}_F)$ on finitely generated $\mathbb{Z}_p$-modules W. This leads us to consider the category of finitely presented representations of G; it also motivates us to prove a version of Theorem 1.2 with coefficients.

I do not know whether Theorem 1.2 holds when $G = GL_n(F)$ (or any p-adic Lie group). The method of this paper does not apply, because G is not (up to centre) an amalgam of two compact open subgroups. I am not sure whether Theorem 1.2 holds when F has positive characteristic; the method of this paper fails because $GL_2(\mathcal{O}_F)$ is not p-adic analytic and its completed group ring is not noetherian. I thank Billy Woods for a helpful discussion about this case.

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2. Finitely presented representations.

For the rest of this article, let $\mathbb{F}$ be a finite field of characteristic p. Let A be a complete local noetherian $W(\mathbb{F})$-algebra with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{F}$. Let G be a locally profinite group. Recall ([Eme10] definition 2.2.5) that a *smooth* A-representation of G is a representation of G on a torsion A-module V such that every $v \in V$ is fixed by a compact open subgroup of G.

Definition 2.1. If K is a profinite group, then a *finite rank* A-representation of K is a representation of K on a finitely generated A-module M such that, for every $n \geq 0$, $M/\mathfrak{m}^n M$ is a smooth representation of K.

[2] The definition of 'finitely presented' in these articles is slightly different to ours, and automatically entails Z-finiteness.

Strictly speaking, we should call these finite rank continuous A-representations of K.

Definition 2.2. A representation of G on an A-module V is K*-finite* if for some (equivalently, any) compact open subgroup $K \subset G$, and for every $v \in V$, the $A[K]$-module generated by v is a finite rank A-representation of K.

We let $\mathcal{C}_A^{K\text{-fin}}(G)$ be the category of all K-finite A-representations of G, with morphisms being morphisms of $A[G]$-modules. Note that a representation of G on a *torsion* A-module V is smooth if and only if it is K-finite.

In the introduction (Definition 1.1) we gave the definitions of 'finitely generated' and 'finitely presented' smooth $\mathbb{F}$-representations of G. We now extend those to K-finite A-representations. First, note that if M is a finite rank A-representation of a compact open subgroup $K \subset G$, then $\operatorname{ind}_K^G M$ is certainly K-finite.

Definition 2.3. Let V be a K-finite A-representation of G. Then V is:

(1) *finitely generated* if there is a compact open subgroup $K \subset G$, a finite rank A-representation W of K, and a surjection of $A[G]$-modules
$$\operatorname{ind}_K^G(W) \twoheadrightarrow V;$$
(2) *finitely presented* if for some compact open subgroups K_1, K_2 of G there is an exact sequence of $A[G]$-modules
$$\operatorname{ind}_{K_1}^G W_1 \to \operatorname{ind}_{K_2}^G W_2 \to V \to 0$$
for W_1 and W_2 finite rank A-representations of K_1 and K_2.

We start by establish some straightforward properties of finitely presented K-finite representations. Many of the proofs follow those of the properties of finitely presented modules over a ring given in [Sta17, Tag 0519].

Lemma 2.4. *A K-finite A-representation V of G is finitely generated if and only if it is finitely generated as an $A[G]$-module.*

Proof. For any W and K, $\operatorname{ind}_K^G W$ is generated (as an $A[G]$-module) by the finitely generated A-submodule of functions supported on K. The 'only if' direction follows.

For the 'if' direction, let V be a K-finite representation generated by $v_1, \ldots, v_n$ as an $A[G]$-module. Choose a compact open subgroup K and let W be the finite rank A-representation of K generated by $v_1, \ldots, v_n$. Then V is a quotient of $\operatorname{ind}_K^G W$. □

Remark 2.5. It is not true that a finitely presented K-finite A-representation of G will be finitely presented as an $A[G]$-module; this is already false for the $\mathbb{F}$-representation $\operatorname{ind}_K^G \mathbb{F}$, as long as K is not finitely generated. This is the main technical problem that we have to overcome in the next section.

Lemma 2.6. *Suppose that $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of K-finite A-representations of G.*

If V_1 and V_3 are finitely generated, so is V_2.

Proof. This is immediate from Lemma 2.4 and the fact that an extension of finitely generated modules over $A[G]$ is finitely generated. □

Lemma 2.7. *Suppose that $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of K-finite A-representations of G.*

(1) *If V_2 is finitely presented and V_1 is finitely generated, then V_3 is finitely presented.*
(2) *If V_3 is finitely presented and V_2 is finitely generated, then V_1 is finitely generated.*
(3) *If V_1 and V_3 are finitely presented, so is V_2.*

Proof. We use K and L, M, N to denote a suitably chosen compact open subgroup of G and finite rank A-representations of K.

(1) Choose a presentation $\operatorname{ind}_K^G N \xrightarrow{\alpha} \operatorname{ind}_K^G M \to V_2 \to 0$ and choose $v_1, \ldots, v_r$ generating the image of V_1 in V_2 as an $A[G]$-module. For each i, let $\tilde{v}_i$ be a lift of v_i to $\operatorname{ind}_K^G M$, and let L be the finite rank A-representation of K generated by the $\tilde{v}_i$. Then we have a map $\gamma : \operatorname{ind}_K^G L \to \operatorname{ind}_K^G M$, and the kernel of the (surjective) composition $\operatorname{ind}_K^G M \to V_2 \to V_3$ is the sum of the image of α and the image of γ, and so is finitely generated.

(2) Choose a presentation $\operatorname{ind}_K^G N \to \operatorname{ind}_K^G M \xrightarrow{\alpha} V_3 \to 0$. We may replace M by its image in V_3, so that we have $M \subset V_3$ and $\operatorname{ind}_K^G M \to V_3$ is the natural map. Let $m_1, \ldots, m_r$ generate $M \subset V_3$ as an A-module, and for each i let $\tilde{m}_i \in V_2$ be a lift of m_i. Let $\tilde{M}$ be the $A[K]$-span of the $\tilde{m}_i$ in V_2. Then there is a surjective map of K representations $\tilde{M} \to M$, and we let L be the kernel. There is also a map $\beta : \operatorname{ind}_K^G \tilde{M} \to V_2$ giving a commuting diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & & & \operatorname{ind}_K^G N & & \\
 & & & & \downarrow & & \\
\operatorname{ind}_K^G L & \longrightarrow & \operatorname{ind}_K^G \tilde{M} & \longrightarrow & \operatorname{ind}_K^G M & \longrightarrow & 0 \\
 & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha} & & \\
 & & V_2 & \longrightarrow & V_3 & \longrightarrow & 0.
\end{array}$$

Repeating the same argument, we may replace N by an $A[K]$-submodule of $\operatorname{ind}_K^G M$ and find a K-submodule $\tilde{N} \subset \operatorname{ind}_K^G \tilde{M}$, together with a surjection $\tilde{N} \to N$ of $A[K]$-modules, such that

$$\begin{array}{ccc}
\operatorname{ind}_K^G \tilde{N} & \longrightarrow & \operatorname{ind}_K^G N \\
\downarrow & & \downarrow \\
\operatorname{ind}_K^G \tilde{M} & \longrightarrow & \operatorname{ind}_K^G M.
\end{array}$$

commutes and has surjective horizontal maps. The kernel of $\operatorname{ind}_K^G \tilde{M} \to V_3$ is the image of $\operatorname{ind}_K^G(\tilde{N} \oplus L)$. Write γ for the restriction of β to $\operatorname{ind}_K^G(\tilde{N} \oplus L)$. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
 & & \operatorname{ind}_K^G(\tilde{N} \oplus L) & \longrightarrow & \operatorname{ind}_K^G \tilde{M} & \longrightarrow & V_3 & \longrightarrow & 0 \\
 & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\beta}\downarrow & & \| & & \\
0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0
\end{array}$$

with exact rows, from which we see that $\operatorname{cok}(\gamma) \cong \operatorname{cok}(\beta)$. As V_2 is finitely generated, so is $\operatorname{cok}(\beta)$ and hence also $\operatorname{cok}(\gamma)$. Since $\operatorname{im}(\gamma)$ is also finitely generated, we see that V_1 is finitely generated by Lemma 2.6.

(3) Choose surjections $\alpha : \operatorname{ind}_K^G M \to V_1$ and $\beta : \operatorname{ind}_K^G N \to V_3$. As before, we may assume that $N \subset V_3$. Let $n_1, \dots, n_r$ generate N as an A-module, lift them to $\tilde{n}_i \in V_2$, and let $\tilde{N}$ be the $A[K]$-module generated by the $\tilde{n}_i$. Let γ be the resulting map $\operatorname{ind}_K^G \tilde{N} \to V_2$. If we let $L = \ker(\tilde{N} \to N)$, then γ restricts to a map $\gamma' : \operatorname{ind}_K^G L \to V_1$. We obtain a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{ind}_K^G(M \oplus L) & \longrightarrow & \operatorname{ind}_K^G(M \oplus \tilde{N}) & \longrightarrow & \operatorname{ind}_K^G N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha+\gamma'} & & \downarrow{\scriptstyle \alpha+\gamma} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0 \end{array}$$

with exact rows and surjective vertical maps. By the snake lemma there is a short exact sequence

$$0 \to \ker(\alpha + \gamma') \to \ker(\alpha + \gamma) \to \ker(\beta) \to 0.$$

Since the outer two terms are finitely generated by (2), so is the inner term (by Lemma 2.6). Thus V_2 is finitely presented, as required. □

Lemma 2.8. *Suppose that $G' \subset G$ is a finite index open subgroup. Then a K-finite A-representation V of G is finitely generated/presented if and only if its restriction to G' is.*

Proof. (1) If V is finitely generated as a representation of G' then it certainly is as a representation of G. Conversely, for any compact open subgroup K of G and any finite rank A-representation W of K, we have the Mackey formula

$$\operatorname{res}_{G'}^G \operatorname{ind}_K^G W \cong \bigoplus_{g \in G' \backslash G / K} \operatorname{ind}_{gKg^{-1} \cap G'}^{G'} W^g.$$

So $\operatorname{ind}_K^G W$ is finitely generated — in fact finitely presented — as a representation of G'. It follows that any finitely generated representation of G is finitely generated as a representation of G'.

(2) We showed in (1) that $\operatorname{ind}_K^G W$ is finitely presented as a representation of G' for any finite rank A-representation W of a compact open subgroup K. It follows from Lemma 2.7 (1) that any K-finite finitely presented representation of G is finitely presented as a representation of G'.

Conversely, suppose that V is finitely presented as a representation of G'. By the first part, it is finitely generated as a representation of G, so that there is a surjection $\operatorname{ind}_K^G W \to V$. Since the first term is finitely generated as a representation of G' by part (1), by Lemma 2.7 (2) the kernel is finitely generated as a representation of G', and hence also as a representation of G. Therefore V is finitely presented as a representation of G by Lemma 2.7 (1). □

2.1. Z**-finiteness.** Suppose that G is a locally profinite group with centre Z. We say that *Hypothesis Z is satisfied* if, for some (equivalently, any) compact open subgroup K of G, $Z/K \cap Z$ is finitely generated. Recall from [Eme10] the definitions of Z-finite and locally Z-finite representations: a representation is Z-finite if the action of $A[Z]$ on V factors through a quotient $A[Z]/I$ that is a finitely generated A-module. It is locally Z-finite if the $A[Z]$-module spanned by any $v \in V$ is a finitely generated A-module. By [Eme10] Lemma 2.3.3, a representation of G, finitely generated as an $A[G]$-module, is Z-finite if and only if it is locally Z-finite.

Lemma 2.9. *Let V be a locally Z-finite, K-finite, A-representation of G.*

(1) *The representation V is finitely generated if and only if there is a surjection*
$$\operatorname{ind}_{KZ}^G W \to V \to 0$$
for some compact open subgroup K of G and finite rank A-representation W of KZ.

(2) *If the representation V is finitely presented then there is an exact sequence*
$$\operatorname{ind}_{K_1 Z}^G W_1 \to \operatorname{ind}_{K_2 Z}^G W_2 \to V \to 0$$
for some compact open subgroup K of G and finite rank A-representations W_1 and W_2 of $K_1 Z$ and $K_2 Z$. If Hypothesis Z is satisfied, the converse holds.

Proof. (1) The backwards implication is clear. For the forwards implication, let W be the $A[KZ]$-span of a finite set of generators of V. It is finite-rank since V is K-finite and locally Z-finite. We therefore get a surjection $\operatorname{ind}_{KZ}^G W \to V \to 0$ as required.

(2) Suppose that V is finitely presented. Then there is a surjection $\operatorname{ind}_{KZ}^G W_2 \to V \to 0$, by the first part. The kernel is finitely generated by Lemma 2.7 (2), and $\operatorname{ind}_{KZ}^G W_2$ is Z-finite. Applying the first part again, we get an exact sequence $\operatorname{ind}_{KZ}^G W_1 \to \operatorname{ind}_{KZ}^G W_2 \to V \to 0$ as required.

For the other direction, it is enough to show that (under Hypothesis Z) $\operatorname{ind}_{KZ}^G W_2$ is finitely presented for any representation W_2 of KZ on a finitely generated A-module. If U is the kernel of the natural map $\operatorname{ind}_K^{KZ} W_2 \to W_2$ then there is a short exact sequence
$$0 \to \operatorname{ind}_{KZ}^G U \to \operatorname{ind}_K^G W_2 \to \operatorname{ind}_{KZ}^G W_2 \to 0.$$
We have to show that U is finitely generated as a KZ-representation. This follows from Hypothesis Z, since this implies that $A[KZ/K]$ is a noetherian ring. □

Now suppose that H is an open subgroup of G such that HZ has finite index in G and $Z \cap H$ is compact.

Proposition 2.10. *Let V be a locally Z-finite, K-finite, A-representation of G.*

(1) *The representation V of G is finitely generated if and only if its restriction to H is finitely generated.*

(2) *If the representation V of G is finitely presented then its restriction to H is finitely presented. If Hypothesis Z holds, then the converse is true.*

Proof. By Lemma 2.8 we may assume that $G = HZ$. Let V be a locally Z-finite K-finite representation of G.

(1) If V is finitely generated as an representation of H, it certainly is as a representation of G. Conversely, suppose that V is finitely generated as an representation of G. If $W \subset V$ is a finitely generated A-module that generates V as an representation of G, then the Z-span ZW is a finitely generated A-module that generates V as a representation of H. So V is a finitely generated representation of H as required.

(2) Suppose that V is finitely presented as a representation of G. By Lemma 2.9 (2) and Lemma 2.7 (1), it suffices to show that $\mathrm{ind}_{KZ}^{G} W$ is a finitely presented representation of H for $K \subset H$. This follows from the identity of representations of H

$$\mathrm{ind}_{KZ}^{HZ} W = \mathrm{ind}_{K(Z\cap H)}^{H} W$$

and the assumption that $Z \cap H$ is compact.

Finally, suppose that V is finitely presented as an representation of H and that Hypothesis Z holds. Then V is finitely generated as a representation of G, so by Lemma 2.9 (1) there is a surjection $\mathrm{ind}_{KZ}^{G} W \to V$. By (1) and Lemma 2.7 (2) the kernel of this map is a finitely generated representation of H. By (1) again, it is a finitely generated representation of G, and so by Lemma 2.9 (1) we have an exact sequence

$$\mathrm{ind}_{KZ}^{G} U \to \mathrm{ind}_{KZ}^{G} W \to V \to 0.$$

As G satisfies Hypothesis Z, by the converse direction of Lemma 2.9 (2), V is a finitely presented representation of G. □

3. Completed group rings.

If K is a profinite group, let

$$A[[K]] = \varprojlim_{J \lhd K \text{open}} A[K/J]$$

be the completed group ring, a compact topological A-algebra.

Lemma 3.1. *Suppose that M is a finite rank A-representation of K. Then there is a unique $A[[K]]$-module structure on M extending the $A[K]$-module structure.*

Proof. For each n, the action of $A[K]$ on $M \otimes_A A/\mathfrak{m}_A^n$ factors through $A[K/J_n]$ for some open subgroup $J_n \subset K$ and so extends uniquely to an action of $A[[K]]$. Since M is finitely generated as an A-module, $M = \varprojlim M \otimes_A A/\mathfrak{m}_A^n$ and the lemma follows. □

Kohlhaase [Koh17] has extended the notion of completed group ring beyond the compact case. Let G be a locally profinite group.

Proposition 3.2 (Kohlhaase)**.** *If $K \subset G$ is a compact open subgroup, then there is a unique A-algebra structure on*

$$A\langle G\rangle = A[G] \otimes_{A[K]} A[[K]]$$

such that the natural maps $A[G] \to A\langle G\rangle$ and $A[[K]] \to A\langle G\rangle$ are A-algebra homomorphisms. This A-algebra is independent of the choice of K up to canonical isomorphism.

Proof. This is shown in section 1 of [Koh17] when A is a field — where what we call $A\langle G\rangle$ is denoted $\Lambda(G)$ — but the proof works verbatim for general rings A. We recall the construction for the reader's convenience. Firstly, if K' is an open subgroup of K, then the natural map of $(A[G], A[[K']])$-bimodules

$$\rho_{K,K'} : A[G] \otimes_{A[K']} A[[K']] \to A[G] \otimes_{A[K]} A[[K]]$$

is an isomorphism.[3] If $K'' \subset K'$ then we have $\rho_{K,K''} = \rho_{K',K''} \circ \rho_{K,K'}$, and so we may construct the direct limit

$$A\langle G\rangle = \varinjlim_{K} \left(A[G] \otimes_{A[K]} A[[K]]\right)$$

which is (canonically) isomorphic to any one of its terms. Now, if $g \in G$ then there is an isomorphism of direct systems

$$\cdot g : A[G] \otimes_{A[K]} A[[K]] \to A[G] \otimes_{A[g^{-1}Kg]} A[[g^{-1}Kg]]$$

taking $h \otimes \kappa$ to $hg \otimes g^{-1}\kappa g$. This defines a right action of G on $A\langle G\rangle$ by left $A[G]$-module isomorphisms, which suffices to define the required ring structure on $A\langle G\rangle$. Precisely, if $h \otimes \kappa \in A[G] \otimes_{A[K]} A[[K]]$ and $h' \otimes \kappa' \in A[G] \otimes_{A[K']} A[[K']]$ are representatives of elements of $A\langle G\rangle$, we may assume that $K \subset h^{-1}K'h$ and define

$$(h' \otimes \kappa')(h \otimes \kappa) = h'h \otimes h^{-1}\kappa' h\kappa \in A[G] \otimes_{A[h^{-1}K'h]} A[[h^{-1}K'h]]. \qquad \square$$

For later use, we record a flatness result:

Lemma 3.3. *The A-algebra $A\langle G\rangle$ is flat as a right $A[[K]]$-module for any compact open subgroup K of G.*

Proof. As in [Koh17], $A\langle G\rangle = A[G] \otimes_{A[K]} A[[K]] \cong \bigoplus_{h \in G/K} A[[K]]$ as right $A[[K]]$-modules, so that $A\langle G\rangle$ is even a free right $A[[K]]$-module. $\square$

Remark 3.4. In the same way we could put an A-algebra structure on $A[[K]] \otimes_{A[K]} A[G]$ (for any compact open subgroup K) and the A-module map $A[[K]] \otimes_{A[K]} A[G] \to A\langle G\rangle$ defined by

$$\kappa \otimes h \mapsto h \otimes h^{-1}\kappa h \in A[G] \otimes_{A[h^{-1}Kh]} A[[h^{-1}Kh]]$$

is an isomorphism of A-algebras. Thus Lemma 3.3 holds with 'right' replaced by 'left'.

Lemma 3.5. *Suppose that V is a K-finite A-representation of G. Then there is a unique $A\langle G\rangle$-module structure on V extending the $A[G]$-module structure.*

Proof. Since V is K-finite, for any compact open subgroup K the action of $A[K]$ extends uniquely to an action of $A[[K]]$ by Lemma 3.1. By the unicity, we have that, for any $h \in G$ and $\kappa \in A[[K]]$, the two actions of $h^{-1}\kappa h$ defined on the one hand by the actions of G and $A[[K]]$, and on the other hand by the action of $A[[h^{-1}Kh]]$, agree. From the formula for multiplication in $A\langle G\rangle$ given in Proposition 3.2, it follows that we can define an action of $A\langle G\rangle$ on V by fixing K and setting

$$(h \otimes \kappa)(v) = h(\kappa(v))$$

for any $h \in A[G]$ and $\kappa \in A[[K]]$, which is clearly the unique action extending those of $A[G]$ and $A[[K]]$. $\square$

Lemma 3.6. *Suppose that V is a K-finite A-representation of G. Then V is finitely generated if and only if it is finitely generated as an $A\langle G\rangle$-module.*

[3] In [Koh17] this is stated for K' normal in K, but it is true for any K' and moreover this is necessary for the construction of the ring structure.

Proof. Suppose that V is finitely generated. By Lemma 2.4, V is finitely generated as an $A[G]$-module, and hence as a $A\langle G\rangle$-module.

Conversely, let V be a K-finite A-representation of G that is finitely generated as a $A\langle G\rangle$-module. Then, if $v_1, \dots, v_r$ generate V and if M is their $A[K]$-span, then M is also preserved by $A[[K]]$ and so

$$V = A\langle G\rangle \cdot M = (A[G] \otimes_{A[K]} A[[K]])M = A[G]M.$$

Therefore V is finitely generated, as required. □

The key technical reason for us to introduce the ring $A\langle G\rangle$ is that it *is* true that a finitely presented K-finite A-representation of G is a finitely presented $A\langle G\rangle$-module — see Remark 2.5. The starting point is the following result of Lazard (see [Eme10] Theorem 2.1.1).

Theorem 3.7. *If G is a p-adic analytic group, then $A[[K]]$ is noetherian for every compact open subgroup K of G.* □

Proposition 3.8. *Suppose that G is a p-adic analytic group. Let V be a K-finite A-representation of G. Then V is finitely presented if and only if it is finitely presented as an $A\langle G\rangle$-module.*

Proof. The backwards implication follows from Lemma 3.6. Suppose that V is finitely presented as an $A\langle G\rangle$-module. Then by Lemma 3.6 there is a surjection

$$\alpha : \mathrm{ind}_K^G W \to V \to 0$$

for some finite rank A-representation W of a compact open subgroup $K \subset G$. The kernel of α is a K-finite representation of G that is finitely generated as an $A\langle G\rangle$-module, by [Sta17, Tag 0519] (5).[4] Therefore it is finitely generated as an A-representation of G, by Lemma 3.6.

Suppose now that V is finitely presented. Then by Lemma 2.7 (2), there is a compact open subgroup K, a finite rank A-representation M of K, and a surjection $\mathrm{ind}_K^G M \to V \to 0$ with finitely generated kernel.

By [Sta17, Tag 0519] (4) and Lemma 3.6, it is enough to show that $\mathrm{ind}_K^G(M)$ is a finitely presented $A\langle G\rangle$-module for the $A\langle G\rangle$-module structure provided by Lemma 3.5. We may think of this instead as the tensor product

$$\mathrm{ind}_K^G(M) \cong A[G] \otimes_{A[K]} M$$

via the isomorphism sending an element $f : G \to M$ of $\mathrm{ind}_K^G(M)$ to $\sum_{g \in G/K} g \otimes f(g^{-1})$. By Lemma 3.1 the action of $A[K]$ on M extends uniquely to one of $A[[K]]$ and we have isomorphisms

$$A\langle G\rangle \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} \otimes A[[K]] \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} M$$

of $A[G]$-modules, and hence of $A\langle G\rangle$-modules (by Lemma 3.5).

Since $A[[K]]$ is noetherian by Theorem 3.7, the finitely generated $A[[K]]$-module M is finitely presented; let $A[[K]]^m \to A[[K]]^n \to M \to 0$ be a presentation. Applying $A\langle G\rangle \otimes_{A[[K]]} -$, we obtain an exact sequence

$$A\langle G\rangle^n \to A\langle G\rangle^m \to A\langle G\rangle \otimes_{A[[K]]} M = A[G] \otimes_{A[K]} M \to 0$$

[4] Strictly speaking, [Sta17, Tag 0519] is only stated for modules over commutative rings. However, it is still true, with an identical proof, in the non-commutative case.

so that $\operatorname{ind}_K^G M = A[G] \otimes_{A[K]} M$ is a finitely presented $A\langle G\rangle$-module, as required. ☐

4. Amalgamations and coherence

Let K_1, K_2 and I be profinite groups equipped with inclusions $f_i : I \hookrightarrow K_i$ of I as a common open subgroup of K_1 and K_2. Then there are maps $f_i : A[[I]] \to A[[K_i]]$ of topological augmented A-algebras.

Let $H = K_1 *_I K_2$ be the amalgamation of K_1 and K_2 along I. By [Ser77], Théorème 1, the natural map $I \to H$ is injective. The following proposition shows that H is naturally a locally profinite topological group:

Proposition 4.1. *With the colimit topology,*[5] *H is a locally profinite group with a basis of open neighbourhoods of the identity being given by open neighbourhoods of I.*

Proof. Let H and H' respectively denote H with the colimit topology and the topology for which translates of open subgroups of I are a basis of open sets. Let $i : H \to H'$ and $j : H' \to H$ be the identity maps; we have to show that they are both continuous. But i is continuous by the universal property of H, and j is continuous because the map $I \to H$ is continuous. ☐

We now consider the amalgamated product of rings, $A[[K_1]] *_{A[[I]]} A[[K_2]]$. Note first that $A[K_1] *_{A[I]} A[K_2]$ is simply the group ring of H over A. This is because the functor $G \mapsto A[G]$ from groups to A-algebras is a left-adjoint, and so commutes with the colimit $*$.

In general, we have A-algebra maps $A[[K_1]] \to A\langle H\rangle$ and $A[[K_2]] \to A\langle H\rangle$ which agree on $A[[I]]$, and so (by the universal property) an A-algebra map $\alpha : A[[K_1]] *_{A[[I]]} A[[K_2]] \to A\langle H\rangle$.

Proposition 4.2. *The map*

$$\alpha : A[[K_1]] *_{A[[I]]} A[[K_2]] \to A\langle H\rangle$$

is an isomorphism of A-algebras.

Proof. Let $R = A[[K_1]] *_{A[[I]]} A[[K_2]]$.

The composite map

$$A[H] = A[K_1] *_{A[I]} A[K_2] \to R \to A\langle H\rangle$$

is easily seen to be the natural map $A[H] \to A\langle H\rangle$. It follows that the image of R in $A\langle H\rangle$ contains $A[H]$ and $A[[K_i]]$ and so in fact is all of $A\langle H\rangle$, whence α is surjective.

Moreover, from the universal property of $\otimes$, for each i we have a map of $(A[H], A[[K_i]])$-bimodules

$$A[H] \otimes_{A[K_i]} A[[K_i]] \to R$$

and these define the *same* map $\beta : A\langle H\rangle \to R$. Since this is a map of right $A[[K_1]]$- and $A[[K_2]]$-modules, we see that $\beta \circ \alpha$ is the identity — it is enough to check that it takes 1 to 1. Therefore α is injective and so an isomorphism. ☐

[5] The coarsest topology on H such that for every topological group G equipped with continuous maps $K_i \to G$ agreeing on I, there is a continuous map $H \to G$ extending these.

4.1. **Coherence.** Recall that a ring R is (left) coherent if any of the following equivalent definitions hold:

(1) every finitely generated left ideal of R is finitely presented;
(2) if $f : M \to N$ is a map of finitely presented left R-modules, then $\ker(f)$ is finitely presented;
(3) the category of finitely presented left R-modules is an abelian subcategory of the category of left R-modules.

Proposition 4.3. *If the rings $A[[K_i]]$ are coherent and $A[[I]]$ is noetherian, then $A\langle H\rangle$ is coherent.*

Proof. This follows immediately from [Å82] Theorem 12; the hypotheses of that theorem are satisfied, by Lemma 3.3. For the convenience of the reader, we summarise the argument of [Å82] in the case of interest to us. It uses the characterisation — due to Chase [Cha60] — of left coherent rings as those for which arbitrary products of right flat modules are flat. Let R, S and T be rings such that S and T are R-algebras, and $Q = S *_R T$ is flat as a right R, S or T-module; we will take $R = A[[I]]$ and $S = A[[K_1]]$, $T = A[[K_2]]$. Then there is a Mayer–Vietoris sequence for Tor^Q in terms of Tor^S, Tor^R and Tor^T. If R is left noetherian and S and T are left coherent, then take a set $(F_i)_{i\in I}$ of right flat Q-modules and compare the Mayer–Vietoris sequence for $\mathrm{Tor}(\prod F_i, M)$ with the product of those for $\mathrm{Tor}(F_i, M)$, for an arbitrary left Q-module M This gives $\mathrm{Tor}_i^Q(\prod F_i, M) = 0$ for $i > 1$. Since S and T are left coherent and that, as R is left noetherian and the F_i are right flat R-modules, $(\prod F_i) \otimes_R M \to \prod(F_i \otimes_R M)$ is injective by [Å82] Lemma 6. It follows that $\mathrm{Tor}_1^Q(\prod F_i, M)$ also vanishes, so that $\mathrm{Tor}_i^Q(\prod F_i, M) = 0$ for all $i > 0$ as required. □

Combining with Theorem 3.7 we get:

Corollary 4.4. *Suppose that H is a p-adic analytic group that is an amalgamated product of two compact open subgroups. Then $A\langle H\rangle$ is coherent.* □

Theorem 4.5. *Suppose that H is a p-adic analytic group that is an amalgamated product of two compact open subgroups. Then the category of finitely presented K-finite A-representations of H is an abelian subcategory of the category of A-representations of H.*

Proof. It suffices to show that the kernel or cokernel of a map of finitely presented K-finite A-representations of H is also a finitely presented K-finite A-representation. This is straightforward for cokernels, and does not require the ring $A\langle H\rangle$. For kernels, suppose that $f : V \to W$ is a map of finitely presented K-finite A-representations of H. Then $\ker(f)$ is a K-finite A-representation of H, and by Proposition 3.8 and Corollary 4.4 it is finitely presented as a left $A\langle H\rangle$-module. By Proposition 3.8 again, it is a finitely presented A-representation of H. □

5. Applications.

Let F be a local field of characteristic 0 with ring of integers $\mathcal{O}_F$ and residue field k of characteristic p, and let D be a division algebra over F with ring of integers $\mathcal{O}_D$. Choose a uniformiser π of D. Let $G = GL_2(D)$ and let $G' = SL_2(D)$ be the subgroup of elements of reduced norm 1. Let $K_1 = GL_2(\mathcal{O}_D)$ and let

$K_1' = SL_2(\mathcal{O}_D) = K' \cap SL_2(D)$. Let $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \in G$, and let $K_2 = \alpha K_1 \alpha^{-1}$ and $K_2' = K_2 \cap G'$. Let

$$I = K_1 \cap K_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_1 : c \equiv 0 \bmod \pi \right\}$$

and $I' = I \cap G' = K_1' \cap K_2'$.

Theorem 5.1. *The category of finitely presented K-finite A-representations of G' is an abelian subcategory of $\mathcal{C}_A^{K\text{-}fin}(G')$.*

Proof. By a theorem of Ihara (Serre [Ser77] Chapter II Corollary 1) we know that $G' = K_1' *_{I'} K_2'$. The theorem follows from Theorem 4.5. □

Corollary 5.2. *The category of finitely presented, K-finite, (locally) Z-finite A-representations of G is an abelian subcategory of $\mathcal{C}_A^{K\text{-}fin}(G)$.*

Proof. Let G^0 be the subgroup of G of elements whose reduced norm is in $\mathcal{O}_F^\times$ and let Z be the centre of G. Then ZG^0 has finite index in G, $Z \cap G^0$ is compact, and $Z/Z \cap K$ is finitely generated for any compact open subgroup K of G. Let $f : V_1 \to V_2$ be a map of K-finite Z-finite finitely presented representations of G. By Proposition 2.10 they are finitely presented representations of G^0. By [Ser77] Chapter II Theorem 3, $G^0 = K_1 *_I K_2$, and so Theorem 4.5 the kernel $\ker(f)$ is finitely presented as a representation of G^0. By Proposition 2.10 again, it is a finitely presented representation of G. □

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