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RESEARCH ARTICLE

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On endomorphism algebras of Gelfand–Graev representations II

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Abstract

Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$ of characteristic p, with Deligne–Lusztig dual G^*. We show that, over $\overline{\mathbb{Z}}[1/pM]$ where M is the product of all bad primes for G, the endomorphism ring of a Gelfand–Graev representation of $G(\mathbb{F}_q)$ is isomorphic to the Grothendieck ring of the category of finite-dimensional $\overline{\mathbb{F}}_q$-representations of $G^*(\mathbb{F}_q)$.

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1 | INTRODUCTION

Let G be a connected reductive group defined over a finite field $\mathbb{F}_q$ of characteristic p, let F be the associated Frobenius endomorphism of G, and let Λ be a subring of $\overline{\mathbb{Q}}$ containing $\overline{\mathbb{Z}}[\frac{1}{p}]$. Let B_0 be an F-stable Borel subgroup of G with (necessarily F-stable) unipotent radical U_0, and let $\psi : U_0^F \longrightarrow \Lambda^\times$ be a regular (also called non-degenerate) character. The Gelfand–Graev representation

$$\Gamma_{G,\psi} := \mathrm{Ind}_{U_0^F}^{G^F} \psi$$

is an important representation of G^F (already studied in [5, section 10] and [7]). Its endomorphism ring

$$\Lambda\mathsf{E}_G := \mathrm{End}_{\Lambda G^F}(\Gamma_{G,\psi})$$

is commutative, independent of the choice of ψ up to isomorphism and, over $\overline{\mathbb{Q}}$, may be identified with the ring of $\overline{\mathbb{Q}}$-valued class functions on $G_{\mathrm{ss}}^{*F^*}$, where (G^*, F^*) is a chosen Deligne–Lusztig dual of (G, F) (see [3]). Such an identification only depends on choices of group homomorphisms $(\mathbb{Q}/\mathbb{Z})_{p'} \simeq \overline{\mathbb{F}}_q^{\times} \hookrightarrow \overline{\mathbb{Q}}^{\times}$, which we fix from now on.

There are then (at least) two natural Λ-lattices in $\overline{\mathbb{Q}}\mathsf{E}_G$: $\Lambda\mathsf{E}_G$ and the lattice $\Lambda\mathsf{K}_{G^*}$ spanned by Brauer characters of irreducible representations of G^{*F^*}; here, K_{G^*} is the Grothendieck ring of the category of finite-dimensional $\overline{\mathbb{F}}_q G^{*F^*}$-modules. Denoting by $G^{*F^*}_{\mathrm{ss}}/\sim$ the set of semi-simple conjugacy classes in G^{*F^*}, we may then, as in [12, section 2.5], identify

$$\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} = \overline{\mathbb{Q}}\mathsf{K}_{G^*} \tag{1.1}$$

as $\overline{\mathbb{Q}}$-algebras, where we recall that the second equality follows from the Brauer character isomorphism $\overline{\mathbb{Q}}\mathsf{K}_{G^*} \xrightarrow{\sim} \overline{\mathbb{Q}}^{G^{*F^*}_{p'}/\sim}$ and from the fact that $G^{*F^*}_{p'}/\sim \; = G^{*F^*}_{\mathrm{ss}}/\sim$. Here $G^{*F^*}_{p'}/\sim$ is the set of p-regular conjugacy classes in G^{*F^*}.

The main result of this paper may now be stated as follows:

Main theorem. *If all bad primes for G are invertible in Λ, then the two Λ-lattices $\Lambda\mathsf{E}_G$ and $\Lambda\mathsf{K}_{G^*}$ of $\overline{\mathbb{Q}}\mathsf{E}_G$ are equal.*

Here, we use the notion of 'bad primes for G' from [17]. Denoting by R the root system of G, a prime number ℓ is called *bad* for G if one of the following three conditions holds: (i) $\ell = 2$, and R has an irreducible factor not of type A; (ii) $\ell = 3$, and R has an irreducible factor of exceptional type (G_2, F_4, E_6, E_7, or E_8); (iii) $\ell = 5$, and R has an irreducible factor of type E_8.

In this theorem, the assumption on the bad primes for G is due to the use of *almost characters* in Lusztig's work on unipotent characters, where bad primes appear in the denominators of the 'Fourier transform matrix'. We expect that the theorem remains true without this assumption, though our present method cannot prove it.

Our theorem improves the equality $\overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{p|W|}]\mathsf{K}_{G^*}$ (where W is the Weyl group of G) in [12, Theorem 2.3] whenever the adjoint group of G is simple of type other than F_4 or G_2 (in these two excluded types, the bad primes and the primes dividing the order of the Weyl group coincide). Moreover, via the $\mathbb{Z}$-model E_G of $\Lambda\mathsf{E}_G$ from [12, section 1.5], if we denote by M is the product of all bad primes for G, then the above theorem implies that $\mathbb{Z}[\frac{1}{pM}]\mathsf{E}_G = \mathbb{Z}[\frac{1}{pM}]\mathsf{K}_{G^*}$. Indeed, this amounts to showing that the identification $\overline{\mathbb{Z}}[\frac{1}{pM}]\mathsf{E}_G = \overline{\mathbb{Z}}[\frac{1}{pM}]\mathsf{K}_{G^*}$ in the above theorem is equivariant under the action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the coefficients, and the proof of this equivariance is the same as that of [12, Corollary 2.4].

Relation with invariant theory

Let $\mathsf{B}_{G^\vee}$ be the ring of functions of the $\mathbb{Z}$-scheme $(T^\vee /\!\!/ W)^{F^\vee}$, where $(G^\vee, T^\vee)$ is the split $\mathbb{Z}$-dual of (G, T) with T an F-stable maximal torus of G, $W = N_{G^\vee}(T^\vee)/T^\vee$ is the Weyl group of $(G^\vee, T^\vee)$, and $F^\vee : T^\vee \longrightarrow T^\vee$ is induced by the action of F on $Y(T^\vee) = X(T)$. If G^* has simply connected derived subgroup, then $\Lambda\mathsf{B}_{G^\vee}$ is also a Λ-lattice of $\overline{\mathbb{Q}}\mathsf{E}_G$ and appears to be significant for the *local Langlands correspondence in families*. Indeed, for GL_n, in the course of constructing this correspondence in joint work with Moss [14], Helm proved in [8, Theorem 10.1] the equality $\Lambda\mathsf{E}_{\mathrm{GL}_n} = \Lambda\mathsf{B}_{\mathrm{GL}_n^\vee}$ for Λ being the ring of Witt vectors of $\overline{\mathbb{F}}_\ell$ with $\ell \neq p$. In our current context (G a connected reductive group over $\mathbb{F}_q$), when G^* has simply connected derived subgroup, it is known that $\mathsf{B}_{G^\vee} = \mathsf{K}_{G^*}$ (see [12, Theorem 3.13]), so that our main theorem yields the equalities

$$\Lambda\mathsf{E}_G = \Lambda\mathsf{K}_{G^*} = \Lambda\mathsf{B}_{G^\vee}$$

for $\Lambda = \overline{\mathbb{Z}}[\frac{1}{pM}]$. In particular, for GL_n, $M = 1$ and so we provide an alternative proof of Helm–Moss's equality.

On the proof of the main theorem

Identify $\Lambda\mathsf{E}_G = e_\psi \Lambda G^F e_\psi \subset \Lambda G^F$ where $e_\psi := \frac{1}{|U_0^F|}\sum_{u\in U_0^F}\psi(u^{-1})u$ is the primitive central idempotent of ΛU_0^F associated to ψ. We may then consider the *symmetrising form*

$$\tau = \tau_G := |U_0^F|\mathrm{ev}_{1_{G^F}} : \Lambda\mathsf{E}_G \longrightarrow \Lambda$$

and denote its $\overline{\mathbb{Q}}$-linear extension again by τ. Here $\mathrm{ev}_{1_{G^F}}$ denotes the evaluation map at 1_{G^F}; recall that a symmetrising form on a finite projective Λ-algebra A is a map $\tau : A \to \Lambda$ such that the map $(a, b) \mapsto \tau(ab)$ is a perfect symmetric bilinear form. It has been shown in [12, Proposition 2.2] that $\tau(\mathsf{K}_{G^*}) \subset \mathbb{Z}$ and that $\tau|_{\Lambda\mathsf{K}_{G^*}} : \Lambda\mathsf{K}_{G^*} \longrightarrow \Lambda$ is a symmetrising form. Therefore, the equality $\Lambda\mathsf{E}_G = \Lambda\mathsf{K}_{G^*}$ will hold if

$$\tau(h\pi) \in \Lambda \text{ for all } h \in \Lambda\mathsf{E}_G \text{ and } \pi \in \Lambda\mathsf{K}_{G^*}. \tag{1.2}$$

Indeed, (1.2) shows that each of $\Lambda\mathsf{E}_G$ and $\Lambda\mathsf{K}_{G^*}$ is contained in the dual of the other with respect to the above bilinear form; as each is self-dual, they are equal.

After preparations on Deligne–Lusztig characters and Curtis homomorphisms (Section 2), we will reduce (1.2) to the study of the condition '$\tau(h\pi) \in \Lambda$' for π the restriction to G^{*F^*} of a (virtual) algebraic $\overline{\mathbb{F}}_q$-representation of G^*, by fitting G^* into a central extension (Section 3) and studying related compatibility questions (Sections 4 and 5). To study the condition '$\tau(h\pi) \in \Lambda$' for such π, we will extend the definition of $\tau(h\pi)$ to $h \in G^F$ (Section 6), reduce the discussion to the case where the semi-simple part s of h is central in G (Section 7), and finally deal with the case of central s (Section 8).

2 | PRELIMINARIES

In this section, we recall some properties of Deligne–Lusztig characters and Curtis homomorphisms that we will need later on.

Deligne–Lusztig characters

Let S be an F-stable maximal torus of G, let P be a Borel subgroup containing S, and let V be the unipotent radical of P. Then we have the Deligne–Lusztig variety (see [6, Definition 9.1.1])

$$DL^G_{S\subset P} = \{gV \in G/V : g^{-1}F(g) \in V \cdot F(V)\},$$

which admits a (left) $G^F \times (S^F)^{\mathrm{op}}$-action. When there is no need to specify the chosen Borel subgroup P, we will write $DL^G_{S\subset P}$ simply as DL^G_S.

We consider the virtual ℓ-adic cohomology $H_c^*(\cdot) = \sum_{j\geqslant 0}(-1)^j H_c^j(\cdot, \overline{\mathbb{Q}}_\ell)$, for ℓ a prime distinct from p. For every character $\chi : S^F \longrightarrow \overline{\mathbb{Q}}^\times$, upon choosing a field embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, we have the corresponding Deligne–Lusztig character

$$R_S^G(\chi)(-) := \mathrm{Tr}\Big(-|H_c^*\big(DL_{S\subset P}^G\big) \otimes_{\overline{\mathbb{Q}}_\ell S^F} \chi\Big) = \frac{1}{|S^F|}\sum_{s\in S^F} \mathrm{Tr}\big((-,s)|H_c^*\big(DL_{S\subset P}^G\big)\big)\chi\big(s^{-1}\big),$$

which is independent of the choice of P and which takes values in $\overline{\mathbb{Q}}_\ell$ *à priori*; but by [5, Proposition 3.3], for any $(g,s) \in G^F \times S^F$, the trace $\mathrm{Tr}((g,s)|H_c^*(DL_{S\subset P}^G))$ is an integer independent of ℓ, so in fact $R_S^G(\chi)$ takes values in $\overline{\mathbb{Q}}$, and it can be verified that $R_S^G(\chi)$ is independent of the choices of ℓ and of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.

Curtis homomorphisms

For an F-stable maximal torus S of G, we consider the Curtis homomorphism

$$\mathrm{Cur}_S^G : \overline{\mathbb{Q}}\mathsf{E}_G \longrightarrow \overline{\mathbb{Q}}S^F$$

defined as in [12, section 1.7] (see also [3, Theorem 4.2]). In terms of the Deligne–Lusztig dual, the map Cur_S^G is simply a 'restriction map to a dual torus': indeed, upon fixing an F^*-stable maximal torus S^* of G^* dual to S (whence a duality $\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F) \simeq S^{*F^*}$ and thus a ring isomorphism $\overline{\mathbb{Q}}S^F \simeq \overline{\mathbb{Q}}^{S^{*F^*}}$), the map Cur_S^G is the unique ring homomorphism making the following diagram commutative (see [12, Lemma 1.6]):

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, "\sim", "(0.1)"'] \arrow[d, "\mathrm{Cur}_S^G"'] & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[d, "\mathrm{Res}"] \\ \overline{\mathbb{Q}}S^F \arrow[r, "\sim"] & \overline{\mathbb{Q}}^{S^{*F^*}} \end{tikzcd} \tag{2.1}$$

We will later need the following formula of Bonnafé–Kessar ([1, Proposition 2.5], with the missing sign factor corrected). For all $h \in \overline{\mathbb{Q}}\mathsf{E}_G \subset \overline{\mathbb{Q}}G^F$,

$$\mathrm{Cur}_S^G(h) = \frac{\epsilon_G\epsilon_S}{|S^F|}\sum_{s\in S^F} \mathrm{Tr}((h,s)|H_c^*(DL_{S\subset P}^G))s^{-1} \in \overline{\mathbb{Q}}S^F. \tag{2.2}$$

Here, as usual, $\epsilon_G = (-1)^{\mathrm{rk}_{\mathbb{F}_q}(G)}$ for G any reductive group over $\mathbb{F}_q$. Observe that (2.2) shows that Cur_S^G is independent of the choice of S^*.

3 | ON CENTRAL EXTENSIONS

For our group G, we can fit its Deligne–Lusztig dual G^* into an F^*-equivariant exact sequence of reductive groups

$$1 \longrightarrow Z^* \longrightarrow H^* \longrightarrow G^* \longrightarrow 1 \tag{3.1}$$

where the derived subgroup of H^* is simply connected and Z^* is a torus central in H^*.

We fix a choice of F-equivariant exact sequence of reductive groups

$$1 \longrightarrow G \longrightarrow H \xrightarrow{\kappa} Z \longrightarrow 1, \tag{3.2}$$

which is dual to (3.1). Let T_H be an F-stable maximal torus of H, let B_H be a Borel subgroup of H containing T_H, and let V be the unipotent radical of B_H. Then

$$DL^H_{T_H \subset B_H} = \bigsqcup_{z \in Z^F} \left(DL^H_{T_H \subset B_H}\right)(z)$$

where for each $z \in Z^F$ we have set

$$\left(DL^H_{T_H \subset B_H}\right)(z) := \left\{hV \in DL^H_{T_H \subset B_H} \ : \kappa(h) = z\right\}.$$

Let $T_G = \ker(\kappa|_{T_H} : T_H \twoheadrightarrow Z)$ (resp., $B_G = \ker(\kappa|_{B_H} : B_H \twoheadrightarrow Z)$), which is an F-stable maximal torus of G (resp., a Borel subgroup of G). Then $T_G \subset B_G$, and the unipotent radical of B_G is also V. As T_G is connected, we have $\kappa(T_H^F) = Z^F$, so for each $z \in Z^F$ we may choose a $\dot{z} \in T_H^F$ such that $\kappa(\dot{z}) = z$. Under the inclusion $G \subset H$, for each $z \in Z^F$ we have

$$DL^H_{T_H \subset B_H}(z) = DL^G_{T_G \subset B_G} \cdot \dot{z} \subset H/V,$$

so that

$$DL^H_{T_H \subset B_H}(z) \simeq DL^G_{T_G \subset B_G} \text{ as } (G^F \times (T_G^F)^{\mathrm{op}})\text{-varieties.}$$

In terms of virtual ℓ-adic cohomology, we therefore have

$$H_c^*\left(DL^H_{T_H \subset B_H}\right) = \sum_{z \in Z^F} H_c^*\left(DL^H_{T_H \subset B_H}(z)\right),$$

and the $H^F \times (T_H^F)^{\mathrm{op}}$-action on $H_c^*(DL^H_{T_H \subset B_H})$ satisfies:

$$\begin{cases} \text{for every } (h,t) \in H^F \times \left(T_H^F\right)^{\mathrm{op}}, (h,t) \cdot H_c^*\left(DL^H_{T_H \subset B_H}(z)\right) \subset H_c^*\left(DL^H_{T_H \subset B_H}(\kappa(ht)z)\right); \\ \text{for every } z \in Z^F, H_c^*\left(DL^H_{T_H \subset B_H}(z)\right) \simeq H_c^*\left(DL^G_{T_G \subset B_G}\right) \text{ as } G^F \times \left(T_G^F\right)^{\mathrm{op}}\text{-modules.} \end{cases}$$

In particular, we obtain the following trace formulae: for $(h,t) \in H^F \times (T_H^F)^{\mathrm{op}}$,

$$\begin{cases} \kappa(ht) \neq 1 \Longrightarrow \mathrm{Tr}\left((h,t) | H_c^*\left(DL^H_{T_H \subset B_H}\right)\right) = 0; \\ (h,t) \in G^F \times \left(T_G^F\right)^{\mathrm{op}} \Longrightarrow \mathrm{Tr}\left((h,t) | H_c^*\left(DL^H_{T_H \subset B_H}\right)\right) = |Z^F| \cdot \mathrm{Tr}\left((h,t) | H_c^*\left(DL^G_{T_G \subset B_G}\right)\right). \end{cases} \tag{3.3}$$

We will later need the compatibility (for $\chi : T_H^F \longrightarrow \overline{\mathbb{Q}}^\times$):

$$R_{T_H}^H(\chi)|_{G^F} = R_{T_G}^G(\chi|_{T_G^F}). \tag{3.4}$$

This follows immediately from the defining formula of $R_{T_H}^H(\chi)$ and (3.3) (see also [6, Proposition 11.3.10]).

4 | A COMPATIBILITY LEMMA

Notation as in Section 3. We extend the F-stable Borel subgroup B_0 of G in Section 1 (used to determine the Gelfand–Graev module $\Gamma_{G,\psi}$) to the F-stable Borel subgroup B_0' of H, so that $B_0'/B_0 = Z$ under (3.2); the unipotent radical of B_0' is then equal to U_0 (the unipotent radical of B_0), and the inclusion $G^F \subset H^F$ induced by (3.2) gives rise to a Λ-algebra inclusion

$$\Lambda\mathsf{E}_G = e_\psi \Lambda G^F e_\psi \hookrightarrow e_\psi \Lambda H^F e_\psi = \Lambda \mathsf{E}_H. \tag{4.1}$$

On the other hand, (3.1) yields the identification

$$(G_{\mathrm{ss}}^{*F^*}/\sim) = \left(H_{\mathrm{ss}}^{*F^*}/\sim\right)/Z^{*F^*}, \tag{4.2}$$

which enables us to regard functions on $G_{\mathrm{ss}}^{*F^*}/\sim$ as functions on $H_{\mathrm{ss}}^{*F^*}/\sim$ which are constant on each Z^{*F^*}-orbit.

Let us prove the following 'compatibility lemma':

Lemma. *The following diagram of rings is commutative:*

$$\begin{tikzcd} \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, "\sim", "(0.1)"'] \arrow[d, hook, "(3.1)"'] & \overline{\mathbb{Q}}^{G_{\mathrm{ss}}^{*F^*}/\sim} \arrow[d, hook, "(3.2)"] \\ \overline{\mathbb{Q}}\mathsf{E}_H \arrow[r, "\sim", "(0.1)"'] & \overline{\mathbb{Q}}^{H_{\mathrm{ss}}^{*F^*}/\sim} \end{tikzcd} \tag{4.3}$$

Proof. Let T_G and T_H be as in Section 3, and choose an F^*-stable maximal torus T_G^* of G^* dual to T_G (resp., T_H^* of H dual to T_H) such that $T_H^*/Z^* = T_G^*$. Then the Weyl groups of (G, T_G), (G^*, T_G^*), (H, T_H) and (H^*, T_H^*) are all the same, and we denote this common Weyl group by W. For each $w \in W$, choose an F-stable maximal torus $T_{G,w}$ of G whose G^F-conjugacy class corresponds to the F-conjugacy class of w in W (with respect to T_G, so that we may choose $T_{G,1} = T_G$); choose $T_{G,w}^* \subset G^*$, $T_{H,w} \subset H$ and $T_{H,w}^* \subset H^*$ in a similar way.

In the toric case where $(G, H) = (T_G, T_H)$, the commutativity of (4.3) follows from toric dualities.

For the general case of (G, H), we use the Curtis embeddings $\mathrm{Cur}^G = (\mathrm{Cur}_{T_{G,w}}^G)_{w\in W}$ and $\mathrm{Cur}^H = (\mathrm{Cur}_{T_{H,w}}^H)_{w\in W}$ (see Section 2) to embed (4.3) into the following cubic diagram of rings:

$$
\begin{tikzcd}
& \prod_{w\in W}\overline{\mathbb{Q}}T^F_{G,w} \arrow[rr, "\sim"] \arrow[dd, hook] & & \prod_{w\in W}\overline{\mathbb{Q}}^{T^{*F^*}_{G,w}} \arrow[dd, hook] \\
\overline{\mathbb{Q}}\mathsf{E}_G \arrow[ur, hook, "\mathrm{Cur}^G"] \arrow[rr, "\sim"] \arrow[dd, hook] & & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} \arrow[ur, hook, "\mathrm{Res}"] \arrow[dd, hook] & \\
& \prod_{w\in W}\overline{\mathbb{Q}}T^F_{H,w} \arrow[rr, "\sim"] & & \prod_{w\in W}\overline{\mathbb{Q}}^{T^{*F^*}_{H,w}} \\
\overline{\mathbb{Q}}\mathsf{E}_H \arrow[ur, hook, "\mathrm{Cur}^H"] \arrow[rr, "\sim"] & & \overline{\mathbb{Q}}^{H^{*F^*}_{\mathrm{ss}}/\sim} \arrow[ur, hook, "\mathrm{Res}"] &
\end{tikzcd}
\tag{4.4}
$$

In (4.4), the right face is clearly commutative; the top and the bottom faces are commutative by (2.1); the back face is the toric case of (4.3) and is hence commutative. So to prove the commutativity of (4.3), it remains to show that the left face in (4.4) is commutative.

Using (2.2) and the relation $\epsilon_H\epsilon_{T_{H,w}} = \epsilon_G\epsilon_{T_{G,w}}$, the commutativity of the left face in (4.4) is equivalent to the property that, for all $h \in \overline{\mathbb{Q}}\mathsf{E}_G \subset \overline{\mathbb{Q}}G^F$ and all $w \in W$, we have

$$\frac{1}{|T^F_{H,w}|}\sum_{t\in T^F_{H,w}} \mathrm{Tr}\Big((h,t)|H^*_c\Big(DL^H_{T_{H,w}}\Big)\Big)t^{-1} = \frac{1}{|T^F_{G,w}|}\sum_{t\in T^F_{G,w}} \mathrm{Tr}\Big((h,t)|H^*_c\Big(DL^G_{T_{G,w}}\Big)\Big)t^{-1}. \tag{4.5}$$

By (3.3) and the fact that $T^F_{H,w}/T^F_{G,w} = Z^F$, we see that (4.5) is true for all $h \in G^F$, so the left face in (4.4) commutes. This completes the proof of the lemma. □

5 | REDUCTION TO THE STUDY OF $\tau(h\pi_\lambda)$

Notation as in Section 3. As Z^{*F^*} is central in H^{*F^*}, the association of each irreducible $\overline{\mathbb{F}}_qH^{*F^*}$-module to its restriction to Z^{*F^*} induces a $\widehat{Z^{*F^*}}$-graded decomposition

$$\mathsf{K}_{H^*} = \bigoplus_{\lambda\in\widehat{Z^{*F^*}}} (\mathsf{K}_{H^*})_\lambda \text{with} \mathsf{K}_{G^*} = (\mathsf{K}_{H^*})_1. \tag{5.1}$$

In particular, we have a ring inclusion $\mathsf{K}_{G^*} \subset \mathsf{K}_{H^*}$, and it is evident that the following diagram of rings is commutative (where br denotes the Brauer character map):

$$
\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[r, "\mathrm{br}", "\sim"'] \arrow[d, hook, "(4.1)"'] & \overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim} \arrow[d, hook, "(3.2)"] \\
\overline{\mathbb{Q}}\mathsf{K}_{H^*} \arrow[r, "\mathrm{br}", "\sim"'] & \overline{\mathbb{Q}}^{H^{*F^*}_{\mathrm{ss}}/\sim}
\end{tikzcd}
\tag{5.2}
$$

Let $h \in \Lambda\mathsf{E}_G$ and $\pi \in \mathsf{K}_{G^*}$. Via the commutative diagrams (4.3) and (5.2), we can define the product $h\pi$ consistently as an element of $\overline{\mathbb{Q}}\mathsf{E}_G$, $\overline{\mathbb{Q}}\mathsf{E}_H$, $\overline{\mathbb{Q}}^{G^{*F^*}_{\mathrm{ss}}/\sim}$, $\overline{\mathbb{Q}}^{H^{*F^*}_{\mathrm{ss}}/\sim}$, $\overline{\mathbb{Q}}\mathsf{K}_{G^*}$ or $\overline{\mathbb{Q}}\mathsf{K}_{H^*}$. As $\tau_G =$

$|U_0^F|\mathrm{ev}_{1_{G^F}} = |U_0^F|\mathrm{ev}_{1_{H^F}} = \tau_H$, we deduce that

$$\tau_G(h\pi) = \tau_H(h\pi). \tag{5.3}$$

Therefore, if we can prove (1.2) for $\tau_H(h\pi)$, then we can prove it for $\tau_G(h\pi)$.

Now let $\mathsf{K}(G^*\text{-mod})$ be the Grothendieck ring of the category of finite-dimensional algebraic G^*-modules and let $\mathsf{K}^\circ_{G^*}$ be the image of the restriction map

$$\mathrm{Res} : \mathsf{K}(G^*\text{-mod}) \longrightarrow \mathsf{K}_{G^*}.$$

Adopting similar notation for H, we have that Res is surjective (see[19, Theorem 7.4] and [9, Theorem 3.10]) so that $\mathsf{K}^\circ_{H^*} = \mathsf{K}_{H^*}$. We are therefore reduced to proving that $\tau_H(h\pi) \in \Lambda$ for $\pi \in \mathsf{K}^\circ_{H^*}$. This turns out to be true without the assumption that H^* has simply connected derived subgroup; in the following, we shall thus return to the group G and study the condition '$\tau_G(h\pi) \in \Lambda$ for $\pi \in \mathsf{K}^\circ_{G^*}$'.

6 | AN EXTENSION $\tilde{\tau}$ FOR $\tau(h\pi)$

We return to the group G (the derived subgroup of G^* may not be simply connected) and write $\tau_G = \tau$. Let T be an F-stable maximal torus of G, let $W = N_G(T)/T$ be the Weyl group of (G, T), and let T_w be an F-stable maximal torus of G associated with $w \in W$ (with respect to T) as in the proof of (4.3). Recall the identification $\overline{\mathbb{Q}}\mathsf{E}_G = \overline{\mathbb{Q}}\mathsf{K}_{G^*}$ from (1.1). Then, for $h \in \overline{\mathbb{Q}}\mathsf{E}_G$ and $\pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}$:

$$\begin{aligned}
\tau(h\pi) &= \frac{1}{|W|}\sum_{w\in W} \mathrm{ev}_{1_{T_w^F}}\left(\mathrm{Cur}^G_{T_w}(h\pi)\right) \quad \text{(by [1, eq. 3.5])}\\
&= \frac{1}{|W|}\sum_{w\in W} \mathrm{ev}_{1_{T_w^F}}\left(\mathrm{Cur}^G_{T_w}(h)\cdot \mathrm{Cur}^G_{T_w}(\pi)\right) \quad (\mathrm{Cur}^G_{T_w} \text{ is a ring homomorphism})\\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{t\in T_w^F} \mathrm{Cur}^G_{T_w}(h)(t^{-1})\cdot \mathrm{Cur}^G_{T_w}(\pi)(t)\\
&= \frac{1}{|W|}\sum_{w\in W}\sum_{t\in T_w^F} \frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\cdot \mathrm{Tr}\Big((h,t)|H_c^*\Big(DL^G_{T_w}\Big)\Big)\cdot \mathrm{Cur}^G_{T_w}(\pi)(t) \quad \text{(by (1.2))}\\
&= \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{t\in T_w^F}\sum_{\chi\in \mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi(t)\cdot \mathrm{Cur}^G_{T_w}(\pi)(t) \quad \text{(trace formula)}\\
&= \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{\chi\in \mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R^G_{T_w}(\chi)(h)\cdot\chi\Big(\mathrm{Cur}^G_{T_w}(\pi)\Big).
\end{aligned} \tag{6.1}$$

Using the formula (5.1), we can extend the $\overline{\mathbb{Q}}$-bilinear map

$$\overline{\mathbb{Q}}\mathsf{E}_G \times \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}, \quad (h,\pi) \longmapsto \tau(h\pi),$$

to a $\overline{\mathbb{Q}}$-bilinear map $\widetilde{\tau}(\cdot,\cdot) : \overline{\mathbb{Q}}G^F \times \overline{\mathbb{Q}}\mathsf{K}_{G^*} \longrightarrow \overline{\mathbb{Q}}$ by setting, for $h \in \overline{\mathbb{Q}}G^F$ and $\pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}$,

$$\widetilde{\tau}(h,\pi) := \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} R_{T_w}^G(\chi)(h)\cdot\chi(\mathrm{Cur}_{T_w}^G(\pi)). \tag{6.2}$$

We then have

$$\tau(h\pi) = \widetilde{\tau}(h,\pi) \text{for all} h \in \overline{\mathbb{Q}}\mathsf{E}_G \text{and all} \pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}. \tag{6.3}$$

The formula (6.2) for $\widetilde{\tau}$ involves choices of T and T_w; we now derive an intrinsic formula for $\widetilde{\tau}$ as follows.

Let $\mathcal{T}_G$ be the set of F-stable maximal tori of G, and let $\mathcal{T}_G/G^F$ be the set of G^F-conjugacy classes in $\mathcal{T}_G$. For each $S \in \mathcal{T}_G$, let $W_G(S) = N_G(S)/S$. As the isomorphism class of T_w depends only on the F-twisted conjugacy class of $w \in W$, and the stabiliser of $w \in W$ under F-twisted conjugacy may be identified with $W_G(T_w)^F$, we have that there are $\frac{|W|}{|W_G(S)^F|}$ elements $w \in W$ such that T_w is G^F-conjugate to S. By (6.2), for $h \in \overline{\mathbb{Q}}G^F$ and $\pi \in \overline{\mathbb{Q}}\mathsf{K}_{G^*}$, we have:

$$\begin{aligned}\widetilde{\tau}(h,\pi) &= \sum_{S\in\mathcal{T}_G/G^F}\frac{\epsilon_G\epsilon_S}{|W_G(S)^F|}\frac{1}{|S^F|}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} R_S^G(\chi)(h)\cdot\chi\big(\mathrm{Cur}_S^G(\pi)\big)\\ &= \frac{1}{|G^F|}\sum_{S\in\mathcal{T}_G}\epsilon_G\epsilon_S\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)} R_S^G(\chi)(h)\cdot\chi\big(\mathrm{Cur}_S^G(\pi)\big).\end{aligned} \tag{6.4}$$

7 | REDUCTION TO THE CASE OF CENTRAL *s*

From now on, let $h = su \in G^F$ with $s \in G^F$ (resp., $u \in G^F$) the semi-simple (resp., unipotent) part in the Jordan decomposition of h. Recall Deligne–Lusztig's character formula [5, Theorem 4.2] for each F-stable maximal torus S of G: (notation: $\mathrm{ad}(g)x = {}^g x = gxg^{-1}$)

$$R_S^G(\chi)(h) = \frac{1}{|C_G(s)^{\circ F}|}\sum_{\substack{g\in G^F\\ g^{-1}sg\in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u)\cdot({}^g\chi)(s), \tag{7.1}$$

where $Q_S^G = R_S^G(\mathbf{1})|_{G^F_{\mathrm{unip}}}$ denotes the Green function and $C_G(s)^\circ$ is the identity component of the centraliser of s in G.

We shall write $\widetilde{\tau} = \widetilde{\tau}_G$ to specify the group G. Substituting (7.1) into (5.4), we obtain: (below, $\pi \in \mathsf{K}_{G^*}$)

$$\begin{aligned}\widetilde{\tau}_G(h,\pi) &= \frac{1}{|G^F|}\sum_{S\in\mathcal{T}_G}\epsilon_G\epsilon_S\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(S^F)}\frac{1}{|C_G(s)^{\circ F}|}\sum_{\substack{g\in G^F\\ g^{-1}sg\in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u)\cdot\chi\Big({}^{g^{-1}}s\cdot\mathrm{Cur}_S^G(\pi)\Big)\\ &= \frac{1}{|G^F|}\frac{1}{|C_G(s)^{\circ F}|}\sum_{S\in\mathcal{T}_G}\epsilon_G\epsilon_S|S^F|\sum_{\substack{g\in G^F\\ g^{-1}sg\in S^F}} Q_{\mathrm{ad}(g)S}^{C_G(s)^\circ}(u)\cdot\mathrm{Cur}_S^G(\pi)\Big({}^{g^{-1}}(s^{-1})\Big)\end{aligned}$$

(where we have applied the orthogonality of characters)

$$= \frac{1}{|G^F|}\frac{1}{|C_G(s)^{\circ F}|}\sum_{g\in G^F}\sum_{\substack{S\in\mathcal{T}_G\\ s\in(\mathrm{ad}(g)S)^F}} \epsilon_G\epsilon_S|S^F|\cdot Q^{C_G(s)^\circ}_{\mathrm{ad}(g)S}(u)\cdot \mathrm{Cur}^G_{\mathrm{ad}(g)S}(\pi)(s^{-1})$$

(where we have used $\mathrm{Cur}^G_{\mathrm{ad}(g)S}(\pi)({}^g x) = \mathrm{Cur}^G_S(\pi)(x)$ for $g\in G^F$)

$$= \frac{1}{|C_G(s)^{\circ F}|}\sum_{\substack{S\in\mathcal{T}_G\\ s\in S^F}} \epsilon_G\epsilon_S|S^F|\cdot Q^{C_G(s)^\circ}_{S}(u)\cdot \mathrm{Cur}^G_{S}(\pi)(s^{-1}) \quad (S\longmapsto \mathrm{ad}(g^{-1})S)$$

$$= \frac{1}{|C_G(s)^{\circ F}|}\sum_{\substack{S\in\mathcal{T}_{C_G(s)^\circ}\\ s\in S^F}} \epsilon_G\epsilon_S|S^F|\cdot Q^{C_G(s)^\circ}_{S}(u)\cdot \mathrm{Cur}^G_{S}(\pi)(s^{-1}), \tag{7.2}$$

where the last equality holds because for $S\in\mathcal{T}_G$, if S^F contains s then $S\subset C_G(s)^\circ$.

Recall the subring $\mathsf{K}^\circ_{G^*}\subset \mathsf{K}_{G^*}$ from Section 5.

Lemma. *Let Λ_0 be a subring of $\overline{\mathbb{Q}}$. Fix an $h = su\in G^F$ as above, and consider the following statement:*

$$\widetilde{\tau}_G(h,\pi)\in\Lambda_0 \quad \textit{for all } \pi\in\mathsf{K}^\circ_{G^*}. \tag{7.3}$$

Suppose that (7.3) is true when G therein is replaced by $C_G(s)^\circ$ (by [6, Proposition 3.5.3], $u\in C_G(s)^\circ$ and hence $h\in C_G(s)^{\circ F}$). Then (7.3) is true for G.

As s is central in $C_G(s)^\circ$, this lemma will reduce the study of the condition (7.3) to the case where s is central in G.

Proof of lemma. First, we require a certain special set of generators for $\mathsf{K}^\circ_{G^*}$. As shown in [10, chapter II.2], for every maximal torus T^* of G^*, the associated formal character map gives a ring isomorphism

$$\mathrm{ch} : \mathsf{K}(G^*\text{-mod}) \xrightarrow{\sim} \mathbb{Z}[X(T^*)]^W,$$

where $X(T^*) = \mathrm{Hom}_{\mathrm{alg}}(T^*,\mathbb{G}_m)$ is the character group of T^* and W is the Weyl group of (G^*,T^*). For $\lambda\in X(T^*)$, set

$$r_{G,\lambda} := \sum_{\mu\in W\lambda}\mu\in\mathbb{Z}[X(T^*)]^W \quad\text{and}\quad \pi_{G,\lambda} := \mathrm{ch}^{-1}(r_{G,\lambda})|_{G^{*F^*}}\in\mathsf{K}^\circ_{G^*},$$

where for $\lambda\in X(T^*)$, $W\lambda$ denotes the W-orbit of λ. Note that the $\mathbb{Z}$-module $\mathbb{Z}[X(T^*)]^W$ is generated by $\{r_{G,\lambda} : \lambda\in X(T^*)\}$, and so the $\pi_{G,\lambda}$ generate $\mathsf{K}^\circ_{G^*}$ as a $\mathbb{Z}$-module.

Choose an F-stable maximal torus T of G containing s, so that T is also an F-stable maximal torus of $C_G(s)^\circ$. To verify (7.3) for the chosen h, it suffices to show that $\widetilde{\tau}_G(h,\pi_{G,\lambda})\in\Lambda_0$ for all $\lambda\in X(T^*)$.

Let S be an F-stable maximal torus of G, choose an F^*-stable maximal torus S^* of G^* dual to S and with a duality $\widehat{\cdot} : S^F \xrightarrow{\sim} \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^{*F^*})$, and fix a choice of $g \in G^*$ such that $S^* = {}^gT^*$. This duality and the fixed embedding $\overline{\mathbb{F}}_q^\times \hookrightarrow \overline{\mathbb{Q}}^\times$ allow us to identify $\overline{\mathbb{Q}}S^F$ with $\overline{\mathbb{Q}}^{S^{*F^*}}$. For each $\mu \in X(T^*)$, set $\mu_{S^*} = {}^g\mu \in X(S^*)$, and define $\phi_S(\mu) \in S^F$ by the relation $\mu_{S^*}|_{S^{*F^*}} = \widehat{\phi_S(\mu)} \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(S^{*F^*})$. We then have a map $\phi_S : X(T^*) \longrightarrow S^F$ that extends to a ring homomorphism

$$\phi_S : \overline{\mathbb{Q}}[X(T^*)] \longrightarrow \overline{\mathbb{Q}}S^F = \overline{\mathbb{Q}}^{S^{*F^*}}.$$

The following diagram then commutes (where $W = W_{G^*}(T^*) = N_{G^*}(T^*)/T^*$):

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{K}(G^*\text{-mod}) \arrow[rr, "\mathrm{ch}", "\sim"'] \arrow[d, "\mathrm{Res}^{G^*}_{G^{*F^*}}"'] & & \overline{\mathbb{Q}}[X(T^*)]^W \arrow[d, "\phi_S"] \\
\overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[r, "\mathrm{br}", "\sim"'] & \overline{\mathbb{Q}}^{G^{*F^*}} \arrow[r, "\mathrm{Res}^{G^{*F^*}}_{S^{*F^*}}"] & \overline{\mathbb{Q}}^{S^{*F^*}}
\end{tikzcd}$$

Combining this with (2.1), we see that the following diagram of rings also commutes:

$$\begin{tikzcd}
\overline{\mathbb{Q}}\mathsf{K}(G^*\text{-mod}) \arrow[r, "\mathrm{Res}^{G^*}_{G^{*F^*}}"] \arrow[d, "\mathrm{ch}", "\wr"'] & \overline{\mathbb{Q}}\mathsf{K}_{G^*} \arrow[r, equal, "(0.1)"] & \overline{\mathbb{Q}}\mathsf{E}_G \arrow[r, "\mathrm{Cur}^G_S"] & \overline{\mathbb{Q}}S^F \\
\overline{\mathbb{Q}}[X(T^*)]^W \arrow[urrr, "\phi_S"'] & & &
\end{tikzcd} \tag{7.4}$$

The commutative diagram (7.4) gives the relation

$$\mathrm{Cur}_S^G(\pi_{G,\lambda}) = \phi_S(r_{G,\lambda}). \tag{7.5}$$

Via the identifications

$$W_{C_G(s)^{\circ*}}(T^*) = W_{C_G(s)^\circ}(T) \leqslant W_G(T) = W_{G^*}(T^*),$$

we may write $W_{G^*}(T^*)\lambda = \bigsqcup_{\lambda' \in \Omega} W_{C_G(s)^{\circ*}}(T^*)\lambda'$ for some finite subset Ω of $W_{G^*}(T^*)\lambda$, so that $r_{G,\lambda} = \sum_{\lambda' \in \Omega} r_{C_G(s)^\circ,\lambda'}$ and then $\mathrm{Cur}_S^G(\pi_{G,\lambda}) = \sum_{\lambda' \in \Omega} \mathrm{Cur}_S^{C_G(s)^\circ}(\pi_{C_G(s)^\circ,\lambda'})$ by (7.5). Applying (6.2) to $\pi = \pi_{G,\lambda}$, we thus deduce that

$$\widetilde{\tau}_G(h, \pi_{G,\lambda}) = \epsilon_G \epsilon_{C_G(s)^\circ} \sum_{\lambda' \in \Omega} \widetilde{\tau}_{C_G(s)^\circ}(h, \pi_{C_G(s)^\circ,\lambda'}). \tag{7.6}$$

By (7.6) and the assumption of the lemma, we get $\widetilde{\tau}_G(h, \pi_{G,\lambda}) \in \Lambda_0$ for all $\lambda \in X(T^*)$, whence $\widetilde{\tau}_G(h, \pi) \in \Lambda_0$ for all $\pi \in \mathsf{K}_{G^*}^\circ$. □

8 | THE CASE OF CENTRAL *s*

Keep the notation T, W and T_w as in Section 6. Let $\pi \in \mathsf{K}_{G^*}$, let $h = su \in G^F$ be as in Section 7, and suppose furthermore that s lies in the centre of G. Then $C_G(s)^\circ = G$, and (7.1) becomes

$R_S^G(\chi)(h) = Q_S^G(u)\chi(s)$, so that (6.2) yields

$$\begin{aligned}\widetilde{\tau}_G(h,\pi) &= \frac{1}{|W|}\sum_{w\in W}\frac{\epsilon_G\epsilon_{T_w}}{|T_w^F|}\sum_{\chi\in\mathrm{Irr}_{\overline{\mathbb{Q}}}(T_w^F)} Q_{T_w}^G(u)\cdot\chi(s)\cdot\chi\left(\mathrm{Cur}_{T_w}^G(\pi)\right)\\ &= \frac{1}{|W|}\sum_{w\in W}\epsilon_G\epsilon_{T_w}Q_{T_w}^G(u)\langle\pi|_{T_w^{*F^*}},\widehat{s^{-1}}\rangle_{T_w^{*F^*}} \quad \text{(orthogonality of characters)}\\ &= \langle\tilde{\pi},\gamma\rangle_{G^{*F^*}} \end{aligned} \tag{8.1}$$

where (using [5, Proposition 7.3])

$$\gamma := \frac{1}{|W|}\sum_{w\in W}\epsilon_G\epsilon_{T_w}Q_{T_w}^G(u)\mathrm{Ind}_{T_w^{*F^*}}^{G^{*F^*}}\widehat{s^{-1}} = \frac{1}{|W|}\sum_{w\in W}Q_{T_w}^G(u)R_{T_w^*}^{G^*}\widehat{s^{-1}}\otimes\mathrm{St}_{G^*}$$

and $\tilde{\pi}$ is any extension of the Brauer character π to an ordinary virtual character (which exists by [15, Theorem 33]). As s lies in the centre of G, $\widehat{s^{-1}}$ is in fact a multiplicative character of G^{*F^*}, so

$$\gamma = \gamma'\otimes\widehat{s^{-1}}\otimes\mathrm{St}_{G^*}$$

with

$$\gamma' := \frac{1}{|W|}\sum_{w\in W}Q_{T_w}^G(u)R_{T_w^*}^{G^*}(\mathbf{1}). \tag{8.2}$$

Our strategy will be to show that γ' is a $\overline{\mathbb{Q}}$-linear combination of irreducible G^{*F^*}-representations with only bad primes appearing in the denominators.

We need some facts from the theory of almost characters, following [13, chapters 3–4]; in the notation of that book, we are considering the case $n = 1$ and L trivial. See also [2, section 7.3] for a concise exposition, but with some extraneous hypotheses. Let c be the order of the automorphism F on W (when G is split, we have $c = 1$); denote by $\widehat{W}_{\mathrm{ex}}$ the set of all $\phi\in\mathrm{Irr}_{\mathbb{Q}}(W)$ that can be extended to a $\mathbb{Q}$-valued irreducible character of $\widehat{W} := W\rtimes\langle F\rangle$ (by [18, Corollary 1.15], every irreducible representation of W over a characteristic 0 field is defined over $\mathbb{Q}$); for each $\phi\in\widehat{W}_{\mathrm{ex}}$, there exists such an extension (in fact, exactly two) $\widetilde{\phi}\in\mathrm{Irr}_{\mathbb{Q}}(\widehat{W})$. Fixing a choice of such $\widetilde{\phi}$, we then call

$$R_{\widetilde{\phi}}^{G^*} := \frac{1}{|W|}\sum_{w\in W}\widetilde{\phi}(wF)R_{T_w^*}^{G^*}(\mathbf{1})$$

an *almost character* of G^{*F^*}.

Recall from Section 1 the definition of bad primes for G. Note that a prime is bad for G if and only if it is bad for G^*. Define

$$M_G = \text{product of all bad primes for } G. \tag{8.3}$$

Using Lusztig's work on unipotent characters,

$$\begin{aligned}&\text{each almost character } R_{\widetilde{\phi}}^{G^*} \text{ is a } \overline{\mathbb{Z}}\left[\tfrac{1}{M_G}\right]\text{-linear combination}\\ &\text{of irreducible } \overline{\mathbb{Q}}\text{-valued unipotent characters of } G^{*F^*}.\end{aligned} \tag{8.4}$$

Indeed, if G^* has connected centre, then [13, Theorem 4.23] expresses $R_{\widetilde{\phi}}$ as a linear combination of unipotent characters of G^{*F^*}. By [13, (4.21.7)], the denominators divide the orders of certain groups $\mathcal{G}_{\mathcal{F}}$ of the form $\prod \mathcal{G}_{\mathcal{F}_i}$ where the product is over the irreducible factors of the root system of G^*. Each $\mathcal{G}_{\mathcal{F}_i}$ is defined in a case-by-case fashion, in a way depending only on the corresponding irreducible factor of the root system, in [13, 4.4–4.13], and has order divisible only by bad primes for that factor. If G^* does not have connected centre, then we choose a short exact sequence

$$1 \to G^* \to H^* \to Z^* \to 1$$

as in (3.2) (with the roles of G^* and G reversed). Extending the chosen maximal F^*-stable torus and Borel from G^* to H^* as in Section 3, we may identify the Weyl groups of G^* and H^*. Using (3.4) (with $\chi = \mathbf{1}$ therein), we then have

$$R_{\widetilde{\phi}}^{H^*}|_{G^{*F^*}} = R_{\widetilde{\phi}}^{G^*},$$

whence $R_{\widetilde{\phi}}^{G^*}$ is $\overline{\mathbb{Z}}[\frac{1}{M_G}]$-linear combination of restrictions to G^{*F^*} of unipotent characters of H^{*F^*}. However, the restriction to G^{*F^*} of a unipotent character of H^{*F^*} is a unipotent character by [6, Proposition 11.3.8], so (8.4) follows.

Now we prove the following lemma:

Lemma. *The sum γ' in (8.2) is a finite $\overline{\mathbb{Z}}[\frac{1}{M_G}]$-linear combination of almost characters of G^{*F^*}.*

Proof. We have

$$\begin{aligned}\gamma' &= \frac{1}{|W|}\sum_{w\in W} R_{T_w}^G(\mathbf{1})(u)R_{T_w^*}^{G^*}(\mathbf{1})\\ &= \frac{1}{|W|}\sum_{w\in W} R_{T_w}^G(\mathbf{1})(u)\sum_{\phi\in\widehat{W}_{\mathrm{ex}}}\widetilde{\phi}(wF)R_{\widetilde{\phi}}^{G^*} \quad \text{(see [2, p. 76])}\\ &= \sum_{\phi\in\widehat{W}_{\mathrm{ex}}} R_{\widetilde{\phi}}^G(u)R_{\widetilde{\phi}}^{G^*};\end{aligned}$$

by (8.4) and the fact that character values of representations of finite groups are algebraic integers, all $R_{\widetilde{\phi}}^G$ must take values in $\overline{\mathbb{Z}}[\frac{1}{M_G}]$. □

Remark. In the above lemma, the class function γ' can in fact be written as a finite $\overline{\mathbb{Z}}$-linear combination of almost characters of G^{*F^*}. We will not need this stronger property of γ' later, so here we only briefly explain how to achieve this, following the complete proof in [11, Remark of Lemma 2.23]. First, one uses a theorem of Shoji ([16, Theorem 5.5]; see also [6, Theorem 13.2.3]) to get that $Q^G_{T_w}(u) = \mathrm{Tr}(wF|H^*_c(\mathcal{B}_u))$ for all $w \in W$, where $\mathcal{B}_u$ is the variety of Borel subgroups of G containing u. One then studies the contribution of each composition factor V of the finite-dimensional $\overline{\mathbb{Q}}_\ell \widehat{W}$-module $H^*_c(\mathcal{B}_u)$ ($\ell \neq p$) to $\mathrm{Tr}(wF|H^*_c(\mathcal{B}_u))$; one proves that $\mathrm{Tr}(wF|V) \neq 0$ only if $V|_W$ is irreducible, and in this case $\mathrm{Tr}(wF|V) = \chi_V(F) \cdot \mathrm{Tr}(wF|\widetilde{\phi})$ for some linear character $\chi_V : \langle F\rangle \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ and some $\widetilde{\phi} \in \mathrm{Irr}_{\mathbb{Q}}(\widehat{W})$ fitting the definition of the almost character $R^{G^*}_{\widetilde{\phi}}$ and on which F^c acts trivially, so that γ' is the sum of finitely many $\chi_V(F) \cdot R^{G^*}_{\widetilde{\phi}}$ with $V|_W$ irreducible. As all eigenvalues of the endomorphism F on $H^*_c(\mathcal{B}_u)$ lie in $\overline{\mathbb{Z}}$ (see [4, Lemma 1.7]), each $\chi_V(F)$ must lie in $\overline{\mathbb{Z}}^\times$, so γ' is a finite $\overline{\mathbb{Z}}$-linear combination of almost characters of G^{*F^*}, as desired.

Using the previous lemma, (7.1), (8.2) and (8.4), we get the following proposition:

Proposition. *We have* $\widetilde{\tau}_G(h,\pi) \in \overline{\mathbb{Z}}[\frac{1}{M_G}]$ *for all* $\pi \in \mathsf{K}_{G^*}$ *and all* $h \in G^F$ *whose semi-simple part* s *is central in* G. *(*M_G *is as in (8.3).)*

End of proof of the main theorem in Section 1

From now on, we remove the assumption that s is central in G.

Observe that a prime number that is bad for $C_G(x)^\circ$ with x a semi-simple element of G^F is also bad for G; indeed, this follows from the definition of bad primes in Section 1 and from the following two facts: (i) if G is simple of type A (resp., of classical type), then the centraliser of every semi-simple element of G has only factors of type A (resp., of classical type); (ii) if G is simple of type G_2, F_4, E_6 or E_7, then the centraliser of every semi-simple element of G cannot have factors of type E_8 (for dimensional reasons).

Therefore, the previous proposition and the lemma in Section 7 together imply that $\widetilde{\tau}_G(h,\pi) \in \overline{\mathbb{Z}}[\frac{1}{M_G}]$ for all $h \in G^F$ and all $\pi \in \mathsf{K}^\circ_{G^*}$. We then deduce from (6.3) that

$$\tau_G(h\pi) = \widetilde{\tau}_G(h,\pi) \in \Lambda\left[\frac{1}{M_G}\right] \text{ for all } h \in \Lambda\mathsf{E}_G \text{ and all } \pi \in \mathsf{K}^\circ_{G^*}. \tag{8.5}$$

Now fit G into the exact sequence (3.2). As H therein has the same type of root datum as G, we have $M_H = M_G$, so (8.5) applied to H gives $\tau_H(h\pi) \in \Lambda[\frac{1}{M_G}]$ for all $h \in \Lambda\mathsf{E}_H$ and all $\pi \in \mathsf{K}^\circ_{H^*} = \mathsf{K}_{H^*}$. For our G, (5.3) then tells us that (1.2) is true when Λ therein is replaced by $\Lambda[\frac{1}{M_G}]$. Consequently, when all bad prime numbers for G are invertible in Λ, we have $\Lambda[\frac{1}{M_G}] = \Lambda$ and $\Lambda\mathsf{E}_G = \Lambda\mathsf{K}_{G^*}$.

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