

IRREDUCIBLE COMPONENTS OF THE MODULI SPACE OF LANGLANDS PARAMETERS

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ABSTRACT. Let F/\mathbb{Q}_p be finite, G be an L -group, and let \mathfrak{X}_G be the moduli space of Langlands parameters $W_F \rightarrow G$, in characteristic distinct from p . First, we determine the irreducible components of \mathfrak{X}_G . Then, we determine the local structure around tamely ramified points for which the image of inertia is regular. This local structure is related to the endomorphism rings of Gelfand–Graev representations, by work of Li. Lastly, we determine an open dense set in \mathfrak{X}_M , when M is a Levi subgroup of G , such that the natural map of moduli stacks $[\mathfrak{X}_M/M^\circ] \rightarrow [\mathfrak{X}_G/G^\circ]$ is smooth on this set.

1. INTRODUCTION

Let p be a prime and let F be a finite extension of \mathbb{Q}_p with Weil group W_F . Let

$$G = G^\circ \rtimes \Gamma$$

with G° a split reductive group scheme¹ over $\mathbb{Z}[1/p]$ and Γ a finite quotient of W_F equipped with an action on G° by automorphisms preserving a split Borel pair (B, T) . Dat, Helm, Kurinczuk, and Moss [DHKM20] have shown that the functor on $\mathbb{Z}[1/p]$ -algebras

$$R \mapsto \{L\text{-homomorphisms } W_F^0 \rightarrow G(R)\}$$

is representable by a scheme \mathfrak{X}_G , locally of finite type and flat over $\mathbb{Z}[1/p]$ — here W_F^0 is a certain choice of discretization of W_F .²

For L a field of characteristic zero, $\mathfrak{X}_{G,L}$ is known to be generically reduced by work of Bellovin–Gee [BG19] and so it has a smooth open dense subset (which may be made explicit). For L of characteristic l , this is far from being true, even for $G = GL_1$. However, one can still hope for a nice description at an open dense set of points. For $G = GL_n$, in the tame case, we did this in [Sho22], finding a local description of \mathfrak{X}_G around a dense subset of its fibre at each prime l . This description turns out to be related to the endomorphism algebra of the (integral) Gelfand–Graev representation, and we applied this to the $l \neq p$ “Breuil–Mézard” conjecture.

Our aim here is to extend the geometric part of that work to general groups G . We do not quite succeed. Roughly, we can only deal with components on which the L -parameter factors through a Levi subgroup in which ‘inertia is regular’. For these components, we obtain a description similar to that for GL_n , again related to

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¹Recall that these are defined to have connected fibres.

²In [DHKM20], they write ${}^L G$ for our G and \hat{G} for our G° . This is more in keeping with the usual notation in the Langlands program, but leads to a proliferation of ‘ L ’s and hats that we prefer to avoid.

the endomorphism algebras of Gelfand-Graev representations by work of Li [Li23]. For $G = GL_n$ this is everything, but for general G there are components which fall outside our description. A limitation is that we restrict to G unramified; however, as in section 5.2 of [DHKM20], in many situations one may reduce to this case. (We also require G° to have smooth centre and simply connected derived subgroup).

The first task, of independent interest, is the determination of the geometrically irreducible components of \mathfrak{X}_G and its reductions modulo primes $l \neq p$. The (geometric) *connected* components of \mathfrak{X}_G , as well as of its base changes to \mathbb{Z}_l (equivalently, \mathbb{F}_l) for primes $l \neq p$, are determined in [DHKM20]. The determination of the irreducible components takes a similar form, and was explained to us some time ago by David Helm; we have written down the argument in section 2 below.

We now outline the contents of this paper in more detail. First, the determination of irreducible components. For G an algebraic group acting on a scheme X , let G_x denote the stabiliser of a point $x \in X(R)$ (it is a closed subscheme of G_R).

Theorem (Corollary 2.7). *Let L be an algebraically closed field of characteristic distinct from p . There is a bijection between the set of irreducible components of $\mathfrak{X}_{G,L}$ and the set of $G^\circ(L)$ -conjugacy classes of pairs (η, α) where:*

- η is a continuous³ L -homomorphism $I_F^0 \rightarrow G(L)$ over L that extends to W_F^0 ; and
- α is a component of the $(G^\circ)_\eta$ -torsor of extensions of η to W_F^0 .

Note that two pairs (η, α) and (η, α') are $G^\circ(L)$ -conjugate if and only if α and α' are conjugate under $\pi_0((G^\circ)_\eta)$.

The critical case (by the ‘tame reduction’ of [DHKM20] sections 3 and 4.5 and the discussion following Lemma 5.7) is when the action of W_F on G° is unramified and η is tamely ramified, so determined by an element $\Sigma \in G^\circ(L)$. In this case the irreducible components of type Σ are in bijection with twisted conjugacy classes of the component group of the centralizer of Σ in G° .

Assume, therefore, that Γ corresponds to an unramified extension of F , and define $\mathfrak{X}_G^{\text{tame}}$ to be the moduli space of *tamely ramified* L -parameters. We have a presentation

$$W_F^0/P_F \cong \langle \text{Fr}, \sigma : \text{Fr} \sigma \text{Fr}^{-1} = \sigma^q \rangle$$

where q is the order of the residue field of F . Define $\mathfrak{X}_G^{\Sigma\text{-reg}}$ to be the open subset of parameters ρ where $\Sigma = \rho(\sigma) \in G^\circ$ is regular. Let T be a maximal split torus in G° preserved by the action of W_F . Let $T//W$ be the GIT quotient of T by the action of the Weyl group W . There is a morphism, the Steinberg morphism,

$$\text{Ch} : G^\circ \rightarrow G^\circ // G^\circ \xrightarrow{\sim} T//W.$$

Let

$$(T//W)^{\text{Fr}^{-1}[q]}$$

be the closed subscheme of $T//W$ fixed by $\text{Fr}^{-1}[q]$, where $[q]$ is the q th power map.

Theorem (Theorem 3.7). *Suppose that G_{der}° is simply connected and that the centre of G° is smooth. Then the morphism*

$$\mathfrak{X}_G^{\Sigma\text{-reg}} \rightarrow (T//W)^{\text{Fr}^{-1}[q]}$$

sending ρ to $\text{Ch}(\rho(\sigma))$ is smooth and surjective.

³For the discrete topology on L .

Remark 1.1. Suppose that G is the L -group of an unramified group with smooth integral model \mathbb{G}/\mathcal{O}_F and that G_{der}° is simply connected. In this case, the coordinate ring of $(T//W)^{\text{Fr}^{-1}[q]}$ has been shown in [Li23] in characteristic not dividing $|W|$, and in [LS] in the case of good characteristic, to be isomorphic to the endomorphism ring of the Gelfand–Graev representation of $\mathbb{G}(k_F)$. This is an inertial shadow of the (conjectural) local Langlands correspondence in families.

Remark 1.2. Under suitable assumptions on G (as in [XZ19] Remark 4.3.4) we might expect a similar result when G is tamely ramified. See Remark 3.9 below.

Remark 1.3. In [DHKM20] Proposition 5.12 one finds conditions under which points of the special fibre with Σ regular unipotent are smooth (at least for $l > h$, where h is the Coxeter number of G). The condition takes the form $l \nmid \chi_G(q)$ where χ_G is a certain product of cyclotomic polynomials depending on G . Our result provides a clean description of \mathfrak{X}_G at such points when $l \mid \chi_G(q)$, as long as G_{der}° is simply connected.

Finally, we can go beyond the Σ -regular locus with the following result, for which we can drop the assumption that G is unramified. By a standard Levi subgroup of ${}^L G$ we mean a subgroup of the form $M = M^\circ \rtimes \Gamma$ where M° is a Γ -stable standard Levi subgroup of G° (relative to our fixed split maximal torus).

Theorem (Theorem 4.2). *Suppose that M is a standard Levi subgroup of G . Consider the morphism*

$$c : G^\circ \times \mathfrak{X}_M \rightarrow \mathfrak{X}_G$$

sending (g, ρ) to $g\rho g^{-1}$. Then there is an open subset $U \subset \mathfrak{X}_M$ such that $c|_U$ is smooth and U intersects each fibre of $\mathfrak{X}_M \rightarrow \text{Spec } \mathbb{Z}[1/p]$ in a dense open subset.

Remark 1.4. The theorem is in the spirit of Lemma 5.14 of [DHKM20], which achieves a similar reduction for their study of unobstructedness.

For $G = GL_n$, the image of

$$(1) \quad \bigcup_M \text{im}(G^\circ \times \mathfrak{X}_M^{\Sigma\text{-reg}} \rightarrow \mathfrak{X}_G^{\text{tame}}),$$

where M runs over conjugacy classes of standard Levi subgroups, is fibrewise dense in \mathfrak{X}_G (see [Sho22]). In general this is not true; there are two issues coming from the disconnectedness of component groups and the existence of non-regular distinguished unipotent conjugacy classes. We give examples in section 5.

In [Sho22] we were able to use this and the description of the coordinate ring of $(T//W)^{[q]}$ as the endomorphism ring of a Gelfand–Graev representation⁴ to give a local proof of the author’s “ $l \neq p$ Breuil–Mézard conjecture” in the tame case. It should be possible to do something similar for unramified groups, but only for modular L -parameters which lie only on components in the image of (1). Given this serious restriction, we will not discuss this point further here.

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⁴In fact, we just need the weaker result after $\otimes \overline{\mathbb{Q}}$.

2. THE MODULI SPACE OF PARAMETERS

2.1. Notation. As in the introduction, let F/\mathbb{Q}_p be a finite extension with ring of integers \mathcal{O}_F and residue field k_F of order q . Let W_F be the Weil group of F with inertia group I_F and wild inertia group P_F . We choose a lift $\text{Fr} \in W_F$ of arithmetic Frobenius and a lift $\sigma \in I_F$ of a topological generator of I_F/P_F in such a way that $\text{Fr} \sigma \text{Fr}^{-1} = \sigma^q$ (this can be done by [Iwa55] Theorem 2). Let W_F^0 be the subgroup of W_F generated by σ, Fr , and P_F ; its intersection with I_F is denoted I_F^0 . We have that $I_F^0/P_F \cong \mathbb{Z}[1/p]$ via an isomorphism sending σ to 1. We topologise W_F^0 such that the cosets of open subsets of P_F (with its profinite topology) form a base for the topology on W_F^0 .

Let G be an algebraic group over $\mathbb{Z}[1/p]$ of the form $G^\circ \rtimes \Gamma$ where: G° is a split reductive group scheme over $\mathbb{Z}[1/p]$, Γ is a finite quotient of W_F , and Γ acts on G° via automorphisms preserving a split Borel pair (B, T) . We write the action of W_F on G° using left superscripts: $(w, g) \mapsto w.g$. Finally, we let $(X^*(T), \Delta, X_*(T), \Delta^\vee)$ be the root datum associated to G° , together with its action of W_F .

If H is a subgroup of W_F and R is a $\mathbb{Z}[1/p]$ -algebra, then an L -homomorphism $H \rightarrow G(R)$ is a continuous (for the discrete topology on $G(R)$) homomorphism

$$\rho : H \rightarrow G(R)$$

such that the composite of ρ with $G(R) \rightarrow \Gamma$ agrees with $H \rightarrow W_F \rightarrow \Gamma$ coming from the given surjection. For $h \in H$, we write $\rho(h) = (\rho^\circ(h), h)$ (slightly loosely using the same letter h for an element of H and its image in Γ). If $H \subset H'$ then by an extension of ρ to H' we will mean an L -homomorphism $\tilde{\rho} : H' \rightarrow G(R)$ whose restriction to H is ρ . If $\gamma \in \Gamma$ then we will write

$$G^\circ \rtimes \gamma = \{(g, \gamma) : g \in G^\circ\}$$

for the corresponding connected component of G .

When G acts on a scheme X and $\eta \in X(S)$, we write G_η for the stabiliser group scheme (defined over S). We adopt the convention that $G_\eta^\circ = (G^\circ)_\eta$, and not $(G_\eta)^\circ$.

If $X \rightarrow S$ and $T \rightarrow S$ are morphisms of schemes, then we will sometimes write $X_T = X \times_S T$ (particularly when T is a point or geometric point of S). If A is a $\mathbb{Z}[1/p]$ -algebra and X is a scheme over $\text{Spec} \mathbb{Z}[1/p]$ then we will write X_A for $X \times_{\text{Spec} \mathbb{Z}[1/p]} \text{Spec} A$, *reserving this notation for this situation*.

2.2. The moduli space of L -parameters. One of the main results of [DHKM20] is the following.

Theorem 2.1 (Dat–Helm–Kurinczuk–Moss). *The functor sending a $\mathbb{Z}[1/p]$ -algebra R to the set of L -homomorphisms $W_F^0 \rightarrow G(R)$ is representable by a scheme \mathfrak{X}_G over $\mathbb{Z}[1/p]$. Moreover, \mathfrak{X}_G has the following properties:*

- (1) *The morphism $\mathfrak{X}_G \rightarrow \text{Spec} \mathbb{Z}[1/p]$ is syntomic (flat, locally of finite presentation, with fibres that are local complete intersections) of relative dimension $\dim_{\mathbb{Z}[1/p]} G$, the relative dimension of G over $\mathbb{Z}[1/p]$.*
- (2) *If L is a field of characteristic zero then $\mathfrak{X}_G \times_{\text{Spec} \mathbb{Z}[1/p]} \text{Spec} L$ is reduced.*⁵
- (3) *The connected components of \mathfrak{X}_G are finitely presented over $\mathbb{Z}[1/p]$.*

If Λ is a $\mathbb{Z}[1/p]$ -algebra, then we write $\mathfrak{X}_{G, \Lambda} = \mathfrak{X}_G \times_{\text{Spec} \mathbb{Z}[1/p]} \text{Spec} \Lambda$. Theorem 2.1 remains true with $\mathbb{Z}[1/p]$ replaced by Λ .

⁵This part relies crucially on prior work of Bellovin-Gee [BG19].

2.3. Components. Let L be an algebraically closed field of characteristic l distinct from p ; we allow $l = 0$. We aim to determine the irreducible components of $\mathfrak{X}_G = \mathfrak{X}_{G,L}$, after first recalling a result on the connected components.

Theorem 2.2 (Dat–Helm–Kurinczuk–Moss). *There are bijections between the sets of:*

- (1) *Connected components of $\mathfrak{X}_{G,L}$; and*
- (2) *$G^\circ(L)$ -conjugacy classes of pairs (η, α) where $\eta : I_F \rightarrow G(L)$ is G -semisimple (see [DHKM20] Definition 4.12) and α is a connected component of the variety of extensions of η to W_F .*

If $l > 0$ then these are in turn in bijection with the sets of connected components of $\mathfrak{X}_{G,\bar{\mathbb{F}}_l}$ and of $\mathfrak{X}_{G,W(\bar{\mathbb{F}}_l)}$.

Proof. The bijection between the first two sets follows from [DHKM20, Theorem 1.7] (see also [DHKM20, Corollary 4.21] parts (2) and (5)).

The final statement follows from [DHKM20, Theorem 4.8]. \square

Remark 2.3. Since η in (5) has finite image (since it is assumed to be continuous for the discrete topology on L), if L has characteristic zero then the G -semisimplicity condition is automatic.

For the rest of this section, we base change everything to L — so $G = G_L$ etc.

Definition 2.4. An L -homomorphism $\eta : I_F^0 \rightarrow G(L)$ is *admissible* if it extends to an L -homomorphism $W_F^0 \rightarrow G(L)$.

Given an admissible η , we consider the affine L -scheme Y_η defined on L -algebras R by

$$Y_\eta(R) = \{\rho : W_F^0 \rightarrow G(R) \text{ extending } \eta\}.$$

We write $\Sigma(\eta)$ for its set of connected components (so we use the same notation as that for a similarly-defined set in [DHKM20] section 3.2).

If we write $\Phi = \rho(\text{Fr})$ then

$$Y_\eta \cong \{\Phi \in G^\circ \rtimes \text{Fr} : \Phi \eta(\gamma) \Phi^{-1} = \eta(\text{Fr } \gamma \text{ Fr}^{-1}) \text{ for all } \gamma \in I_F^0\}$$

and we see that Y_η is a left G_η° -torsor via left multiplication on Φ . If we fix ρ_0 extending η with $\rho_0(\text{Fr}) = \Phi_0$, then we can identify $Y_\eta \cong G_\eta^\circ$, with a general point having $\rho(\text{Fr}) = \phi \Phi_0$ for $\phi \in G_\eta^\circ$.

There is a second action of G_η° on Y_η given by conjugation. On choosing ρ_0 as in the previous paragraph, this may be identified with the action of G_η° on itself by Ad_{Φ_0} -twisted conjugation.

If \mathfrak{J} is the moduli space of L -homomorphisms $I_F^0 \rightarrow G(R)$ (for L -algebras R), then \mathfrak{J} is a union of connected components that are each affine of finite type over L . The G° -orbit of $\eta \in \mathfrak{J}(L)$, $G^\circ \cdot \eta$, then naturally has the structure of a quasiaffine variety and, by [Mil17] Corollary 7.13, there is an isomorphism

$$G^\circ / G_\eta^\circ \xrightarrow{\sim} G^\circ \cdot \eta.$$

There is a natural morphism $p_I : \mathfrak{X}_G \rightarrow \mathfrak{J}$ and we write

$$\mathfrak{X}_\eta = p_I^{-1}(G^\circ \cdot \eta),$$

a locally closed subscheme of \mathfrak{X}_η . Note that, by definition, we have

$$Y_\eta = p_I^{-1}(\eta).$$

Theorem 2.5 and Corollary 2.7 below were explained to the author by David Helm.

Theorem 2.5. *For each admissible L -homomorphism $\eta : I_F^0 \rightarrow G(L)$, every connected component of \mathfrak{X}_η is irreducible, and its Zariski closure in \mathfrak{X}_G is an irreducible component of \mathfrak{X}_G .*

If η is chosen as above, then there is a bijection between:

- (1) *The set of connected (or irreducible) components of \mathfrak{X}_η ;*
- (2) *The set of $\pi_0(G_\eta^\circ)$ -conjugacy classes in $\Sigma(\eta)$.*

If we choose a $\rho_0 : W_F^0 \rightarrow G(L)$ extending η , then these are in bijection with the $\text{Ad}_{\rho_0(\text{Fr})}$ -twisted conjugacy classes in $\pi_0(G_\eta^\circ)$.

Remark 2.6. From the proof of the theorem we will see that $|\mathfrak{X}_\eta|$ is an open subset of an irreducible component of $|\mathfrak{X}_G|$. However, if $\text{char } L \neq 0$, then neither \mathfrak{X}_η nor \mathfrak{X}_G need be reduced, and the non-reduced structures will typically not agree.

Proof. Note that we have an isomorphism

$$\begin{aligned} Y_\eta \times G^\circ &\cong \mathfrak{X}_\eta \times_{G^\circ \cdot \eta} G^\circ \\ (\rho, g) &\mapsto (g\rho g^{-1}, g) \end{aligned}$$

by definition. Taking the quotient by G_η° and using that $G^\circ/G_\eta^\circ \cong G^\circ \cdot \eta$, we obtain an isomorphism

$$\mathfrak{X}_\eta \cong G^\circ \times_{G_\eta^\circ} Y_\eta.$$

It follows that \mathfrak{X}_η is a disjoint union of irreducible connected components corresponding to the G_η° -orbits in $\pi_0(Y_\eta) = \Sigma(\eta)$, which are the same as the $\pi_0(G_\eta^\circ)$ -orbits.

Each component of Y_η has dimension $\dim G_\eta^\circ$ and so the dimension of every component of \mathfrak{X}_η is

$$\dim G^\circ + \dim G_\eta^\circ - \dim G_\eta^\circ = \dim G^\circ = \dim \mathfrak{X}_G.$$

Since $G^\circ \cdot \eta$ is locally closed, the same is true for \mathfrak{X}_η . It follows that the underlying topological space of each connected component of \mathfrak{X}_η is an open subset of an irreducible component of \mathfrak{X}_η . \square

Corollary 2.7. *There are bijections between:*

- (1) *The set of irreducible components of $\mathfrak{X}_{G,L}$;*
- (2) *The set of $G^\circ(L)$ -conjugacy classes of pairs (η, C) where $\eta : I_F^0 \rightarrow G(L)$ is an admissible L -homomorphism and C is a connected component of $\Sigma(\eta)$.*

Let \mathcal{T} be a set of representatives of the $G^\circ(L)$ -conjugacy classes η as in (2). For each $\eta \in \mathcal{T}$, choose an extension ρ_η of η to W_F^0 .

Then these sets are in bijection with the set

$$\{(\eta, C) : \eta \in \mathcal{T} \text{ and } C \text{ is an } \text{Ad}_{\rho_\eta(\text{Fr})}\text{-twisted conjugacy class of } \pi_0(G_\eta^\circ)\}.$$

Proof. By Theorem 2.5, it is enough to prove that every irreducible component of \mathfrak{X}_G arises as the closure of an irreducible component of some \mathfrak{X}_η , for η unique up to G° -conjugacy.

To this end, for each irreducible component \mathcal{C} of \mathfrak{X}_G , let ρ be an L -point that lies only on that component and take $\eta = \rho|_{I_F^0}$. Then one of the components of \mathfrak{X}_η contains ρ , and its closure is an irreducible component of \mathfrak{X}_G containing ρ , which must therefore be equal to \mathcal{C} .

Finally, to see that η is unique, note that if \mathcal{C} is the closure of a connected component of \mathfrak{X}_η then it has an open dense subset of points for which the restriction to I_F^0 is G° -conjugate to η . This can clearly happen for only one G° -conjugacy class of η . \square

3. THE Σ -REGULAR LOCUS

3.1. Regular elements and the Steinberg morphism. *For this subsection only*, let G be a split connected reductive group defined over a ring Λ . Let T be a split maximal torus and $B \supset T$ a Borel subgroup. Let W be the Weyl group of G relative to T . Let r be the rank of T .

Consider the conjugation action of G on itself, and denote the GIT quotient $\text{Spec } \Lambda[G]^G$ by $G//G$. We also define

$$T//W = \text{Spec } \Lambda[T]^W = \text{Spec } \Lambda[X^*(T)]^W.$$

Theorem 3.1. (1) *The restriction morphism*

$$T//W \rightarrow G//G$$

is an isomorphism.

(2) *The formation of $T//W$ and $G//G$ is compatible with base change: for any Λ -algebra Λ' ,*

$$G_{\Lambda'}//G_{\Lambda'} \cong (G//G)_{\Lambda'}$$

and similarly for $T//W$.

Proof. (1) Over a field this is a theorem of [Ste65]. Over a general ring see [Lee15].

(2) This is [Lee15, Lemma 4.2] for $G//G$, and [Lee15, Lemma 3.1] for $T//W$. \square

If L is a field (and a Λ -algebra) then, for any $g \in G(L)$, its centraliser G_g has dimension $\geq r$, and we say that g is regular if equality holds. Then

$$G^{\text{reg}} = \{g \in G : g \text{ is regular in } G(\kappa(g))\}$$

is an open subset of G (by semicontinuity, [Ber65, Proposition 4.1]).

Theorem 3.2. *Suppose that G_{der} is simply connected. Then*

(1) *The natural morphism*

$$\text{Ch} : G \rightarrow G//G$$

is faithfully flat, and its restriction to G^{reg} is smooth and surjective.

(2) *There is an isomorphism*

$$G//G \cong \mathbb{A}_\Lambda^r \times G/G_{\text{der}}.$$

Proof. (1) By the fibrewise criterion for flatness, we may check this after base-change to a field. It then follows from [BR22, Propositions 2.4 and 2.5].

(2) This is [BR22, Lemma 2.3] over a field; however, the proof works over Λ if one takes χ_α in that proof to be the character of $\text{Ind}_B^G \omega_\alpha$ rather than its socle. (In the semisimple simply connected case, see also [Lee15, Corollary 5.2]). \square

Finally, we recall the statement of [Cot22, Theorem 7.13] (noting that, if G_{der} is simply connected, then ‘regular’ and ‘strongly regular’ in [Cot22] are equivalent).

Proposition 3.3. *(Cotner) Suppose that $\Lambda = L$ is a field, that G_{der} is simply connected, and that Z_G is smooth. Then, for every regular element $g \in G(L)$, the centralizer G_g is smooth.*

3.2. Application to \mathfrak{X}_G . Return to the notation of section 2.1. For this subsection we assume that G is *unramified*, in the sense that I_F acts trivially on G° . We then may assume that $I_F \subset \ker(W_F \rightarrow \Gamma)$, and consider only *tame* parameters: those that factor through W_F^0/P_F . Define $\mathfrak{X}_G^{\text{tame}}$ to represent the functor on Λ -algebras

$$\mathfrak{X}_G^{\text{tame}}(R) = \{L\text{-parameters } \rho : W_F^0/P_F \rightarrow G(R)\}.$$

Remark 3.4. By [DHKM20] (4.4) and (4.5), the critical case to consider is that of tame parameters for tamely ramified G — and when considering $\mathfrak{X}_{G, \mathbb{Z}_l}^{\text{tame}}$ it is possible to assume that the order of the automorphism σ is a power of l . In particular, if G is semisimple and simply connected, then it is no great loss to assume that G is unramified so long as $l > 3$.

We have that

$$W_F^0/P_F = \langle \sigma, \text{Fr} : \text{Fr} \sigma \text{Fr}^{-1} = \sigma^q \rangle$$

and, for a parameter $\rho : W_F^0 \rightarrow G(R)$, we write

$$\rho(\sigma) = \Sigma_\rho \in G^\circ$$

(recalling that G is unramified) and

$$\rho(\text{Fr}) = (\Phi_\rho, \text{Fr}).$$

Let p_Σ and p_Φ be the morphisms $\mathfrak{X}_G^{\text{tame}} \rightarrow G^\circ$ taking ρ to Σ_ρ and Φ_ρ respectively.

Let $A_G = G^\circ // G^\circ \cong T // W$. By Theorem 3.1 we have a morphism

$$\text{Ch} : G^\circ \rightarrow A_G.$$

Note that Fr acts on W , T , and A_G . Let $[q] : G^\circ \rightarrow G^\circ$ be the q th power map.

The morphisms Fr and $[q]$ each descend to morphisms, also denoted Fr and $[q]$, from $G^\circ // G^\circ$ to $G^\circ // G^\circ$. We let

$$B_G = (G^\circ // G^\circ)^{\text{Fr}^{-1}[q]} = A_G^{\text{Fr}^{-1}[q]}$$

be the fixed-point subscheme of A_G under the map $\text{Fr}^{-1}[q]$.

Lemma 3.5. *The morphism $\mathfrak{X}_G^{\text{tame}} \rightarrow A_G$ sending ρ to $\text{Ch}(\Sigma_\rho)$ factors through a map*

$$\text{Ch}_\Sigma : \mathfrak{X}_G^{\text{tame}} \rightarrow B_G.$$

Proof. If ρ is an R -point of $\mathfrak{X}_G^{\text{tame}}$ for a $\mathbb{Z}[1/p]$ -algebra R , then $\rho(\text{Fr})\Sigma_\rho\rho(\text{Fr})^{-1} = \Sigma_\rho^q$. This is equivalent to

$$\Phi_\rho^{\text{Fr}\Sigma_\rho}\Phi_\rho^{-1} = \Sigma_\rho^q$$

and so

$$\text{Fr}\text{Ch}(\Sigma_\rho) = \text{Ch}(\text{Fr}\Sigma_\rho) = \text{Ch}(\Sigma_\rho^q) = [q]\text{Ch}(\Sigma_\rho).$$

Thus $\text{Ch}(\Sigma_\rho) \in (G^\circ // G^\circ)^{\text{Fr}^{-1}[q]} = B_G$ as required. \square

Remark 3.6. Suppose that Λ is a $\mathbb{Z}[1/p]$ -algebra and let $A_{G,\Lambda} = T_\Lambda//W$ and $B_{G,\Lambda} = A_{G,\Lambda}^{\text{Fr}^{-1}[q]}$. By Theorem 3.1 part (2), $A_{G,\Lambda} = A_G \times_{\text{Spec } \mathbb{Z}[1/p]} \text{Spec } \Lambda$. It follows formally that $B_{G,\Lambda} = B_G \times_{\text{Spec } \mathbb{Z}[1/p]} \Lambda$. Finally, the map $\text{Ch}_\Lambda : G_\Lambda^\circ \rightarrow A_{G,\Lambda}$ provided by Theorem 3.1 part (1) agrees with the base change of Ch to Λ , and Lemma 3.5 holds over Λ .

Let $\mathfrak{X}_G^{\Sigma\text{-reg}} = p_\Sigma^{-1}(G^{\circ,\text{reg}})$. It is an open subscheme of $\mathfrak{X}_G^{\text{tame}}$.

Theorem 3.7. *Suppose that G_{der}° is simply connected and that Λ is a $\mathbb{Z}[1/p]$ -algebra such that $Z_{G^\circ,\Lambda}$ is smooth over Λ . Then the restriction*

$$\text{Ch}_{\Sigma,\Lambda}|_{\mathfrak{X}_{G,\Lambda}^{\Sigma\text{-reg}}} : \mathfrak{X}_{G,\Lambda}^{\Sigma\text{-reg}} \rightarrow B_{G,\Lambda}$$

is smooth and surjective.

Proof. Let

$$\begin{aligned} \mathfrak{Z}^{\text{reg}} &= \{\Sigma \in G^{\circ,\text{reg}} : \Sigma^q \in G^{\circ,\text{reg}} \text{ and } \text{Ch}(\text{Fr}\Sigma) = \text{Ch}(\Sigma^q)\} \\ &\subset G^{\circ,\text{reg}} \times_{A_G} B_G \end{aligned}$$

with the latter inclusion being an open immersion. We show that in fact it is an equality. For this, it suffices to show that, if $\Sigma \in G^\circ(\kappa)$ is regular, with κ an algebraically closed field (and Λ -algebra), such that $\text{Ch}(\text{Fr}\Sigma)$ and $\text{Ch}(\Sigma)$ are equal, then $\text{Fr}\Sigma$ is conjugate to Σ^q . Considering the su decomposition of Σ and replacing G by $Z_G(s)^\circ$, it is enough to prove that if Σ is regular unipotent then so are $\text{Fr}\Sigma$ and Σ^q . The first is clear. For the second, we may assume that G is semisimple and use the characterisation of regular unipotent elements as those whose projections onto all the root spaces for simple roots are nontrivial (we thank Sean Cotner for help with this).

Then the map $\text{Ch} : \mathfrak{Z}^{\text{reg}} \rightarrow B_G$ is smooth and surjective by Theorem 3.2 (1), and $p_\Sigma|_{\mathfrak{X}^{\Sigma\text{-reg}}}$ factors through a map

$$p_\Sigma : \mathfrak{X}^{\Sigma\text{-reg}} \rightarrow \mathfrak{Z}^{\text{reg}}$$

that we wish to show is smooth.

Note that there is a closed immersion

$$\begin{aligned} \iota : \mathfrak{Z}^{\text{reg}} &\rightarrow G^{\circ,\text{reg}} \times_{A_G} G^{\circ,\text{reg}} \\ \Sigma &\mapsto (\text{Fr}\Sigma, \Sigma^q). \end{aligned}$$

If we let

$$c : G^\circ \times G^{\circ,\text{reg}} \rightarrow G^{\circ,\text{reg}} \times_{A_G} G^{\circ,\text{reg}}$$

be the conjugation morphism $c(\gamma, g) = (g, \gamma g \gamma^{-1})$ then the pullback of $\mathfrak{Z}^{\text{reg}}$ along c is

$$\begin{aligned} &\{(\gamma, g, \Sigma) \in G^\circ \times G^{\circ,\text{reg}} \times G^{\circ,\text{reg}} : g = \text{Fr}\Sigma, \gamma g \gamma^{-1} = \Sigma^q\} \\ &= \{(\gamma, \Sigma) : \gamma^{\text{Fr}} \Sigma \gamma^{-1} = \Sigma^q\} \\ &= \mathfrak{X}_G^{\Sigma\text{-reg}}. \end{aligned}$$

It therefore suffices to show that c is smooth and surjective after base change to Λ . We think of c as a morphism of A_G -schemes where $G^\circ \times G^{\circ,\text{reg}}$ is regarded as an A_G -scheme via Ch on the second factor. Since $G \times G^{\circ,\text{reg}}$ is flat over A_G by

Theorem 3.2 (1), we may apply the fibrewise criterion. It is therefore enough to show that, for every geometric point $s : \text{Spec } \kappa \rightarrow A_{G,\Lambda}$,

$$c_s : G_\kappa^\circ \times_{\text{Spec } \Lambda} G_s^{\circ,\text{reg}} \rightarrow G_s^{\circ,\text{reg}} \times_{\text{Spec } \Lambda} G_s^{\circ,\text{reg}}$$

is smooth and surjective.

By Theorem 3.2 (1), $G_s^{\circ,\text{reg}}$ is smooth and by [BR22] Proposition 2.5 it is a single G_κ° -orbit. The fibre of c_s above a point $(g, g') \in G_s^{\circ,\text{reg}}(\kappa) \times G_s^{\circ,\text{reg}}(\kappa)$ is then a $G_{\kappa,g}^\circ$ -torsor. It follows from the miracle flatness theorem that c_s is flat, and from Proposition 3.3 that it is smooth. \square

As an application, we generalise [Hel20] Proposition 5.3.

Corollary 3.8. *Suppose that the hypotheses of Theorem 3.7 hold with $\Lambda = \mathbb{Z}_l$. Let $S = \mathcal{O}(\mathfrak{X}_{G,\Lambda}^{\text{tame}})^{G_\Lambda^\circ}$ and $R = \mathcal{O}(B_{G,\Lambda})$. Then the natural map $\text{Ch}_\Sigma^* : R \rightarrow S$ is injective with saturated image.*

Proof. Injectivity follows from the faithful flatness of Ch_Σ restricted to the regular locus. If $l \neq p$ is a prime, then the same argument shows that $\text{Ch}_\Sigma^* \otimes \mathbb{F}_l = \text{Ch}_{\Sigma, \mathbb{F}_l}^*$ (see Remark 3.6) is injective. Since S is a torsion-free Λ -module, this implies that the image of Ch_Σ^* is saturated. \square

Remark 3.9. Remark 3.4 notwithstanding, it would be desirable to generalise these results to the case that G is tamely ramified. In this case $G^\circ // G^\circ$ should be replaced by

$$A_{G,\sigma} = (G^\circ \rtimes \sigma) // G^\circ = G^\circ //_\sigma G^\circ$$

(where $//_\sigma$ denotes the quotient for G° acting by σ -twisted conjugation). The q -power map and Frobenius map define G° -equivariant morphisms

$$G^\circ \rtimes \sigma \rightarrow G^\circ \rtimes \sigma^q$$

and we let $B_{G,\sigma} = A_{G,\sigma}^{\text{Fr}^{-1}[q]}$ be their equaliser. There is a notion of σ -regularity, for which see [XZ19] section 5. We then expect that, under appropriate hypotheses on G° (perhaps those of [XZ19] Remark 4.3.4 together with a hypothesis on the centre) there will be a smooth morphism $\mathfrak{X}_G^{\Sigma\text{-reg}} \rightarrow B_{G,\sigma}$. For this we would require generalizations of the results of section 3.1: that is, the results of [XZ19] sections 4 and 5 over a general base and for a reductive (rather than semisimple) group, and Proposition 3.3 in the twisted context.

Remark 3.10. If G is the L -group of an unramified group then we have a representation-theoretic interpretation of B_G , see Remark 1.1. It would be interesting to have a similar representation-theoretic interpretation of $B_{G,\sigma}$, as defined in Remark 3.9, in the tamely ramified case.

4. LEVI SUBGROUPS

For this section we no longer assume that G is unramified. Let $M \subset G$ be a standard Levi subgroup in the sense of [Bor79, section 3]: that is, a subgroup of the form $M^\circ \rtimes \Gamma$ where M° is a Γ -stable standard (with respect to our chosen T) Levi subgroup of G° . Since Γ preserves a Borel subgroup $B \supset T$, there is a Γ -stable standard parabolic subgroup P° with Levi M° , and we write $P = P^\circ \rtimes \Gamma$. Let U be the unipotent radical of P° and U^- its opposite. We have a decomposition

$\mathrm{Lie}(G) = \mathrm{Lie}(M) \oplus \mathrm{Lie}(U) \oplus \mathrm{Lie}(U^-)$ that is stable under the action of W_F^0 on $\mathrm{Lie}(G)$. We also have a natural closed immersion

$$\mathfrak{X}_M \hookrightarrow \mathfrak{X}_G$$

and a conjugation map

$$\begin{aligned} c : G^\circ \times \mathfrak{X}_M &\rightarrow \mathfrak{X}_G \\ (g, \rho) &\mapsto g\rho g^{-1}. \end{aligned}$$

We will show that there is an open and fibrewise dense subset $V \subset \mathfrak{X}_M$ such that $c : G^\circ \times V \rightarrow \mathfrak{X}_G$ is smooth. If we fix a separated filtration $(P_F^d)_{d \geq 0}$ of P_F by open subgroups then

$$\mathfrak{X}_M^d = \{\rho \in \mathfrak{X}_M : \rho|_{P_F^d} \text{ is trivial}\}$$

is a union of connected components of \mathfrak{X}_M .

Let $d \geq 1$ and let r be an integer, depending on d , such that:

- $\mathrm{Fr}^r = e$ in Γ ;
- The conjugation action of Fr^r on P_F/P_F^d is trivial; and
- For any semisimple $s \in G^\circ(\overline{\mathbb{Q}}) \rtimes \sigma$ such that s is conjugate to s^q , we have $s = s^{q^r}$.

Such an integer must exist; for instance, choose a faithful representation $G \hookrightarrow GL_N$ and take $r = n! \mathrm{ord}_\Gamma(\mathrm{Fr}) |\mathrm{Aut}(P_F/P_F^d)|$. The first two conditions immediately follow from the divisibility of r by its second and third factors, respectively. The third condition follows from the divisibility of r by $n!$ and the fact that, if $s \in GL_N(\overline{\mathbb{Q}})$ with s conjugate to s^q , then $s = s^{q^{n!}}$. See also [DHKM20] Lemma 2.2 for a similar argument. Say that $m \in M^\circ(R) \rtimes \mathrm{Fr}$ is *avoidant at depth d* if:

- $\mathrm{ad}_m - 1$ and $\mathrm{ad}_m - q$ are invertible on $\mathrm{Lie}(U)(R)$ and $\mathrm{Lie}(U^-)(R)$;
- if χ is the characteristic polynomial of ad_{m^r} on $\mathrm{Lie}(M)(R) \oplus \mathrm{Lie}(U)(R)$, and $\gamma \in \Gamma$, then $\chi(\mathrm{ad}_{\gamma m^r})$ is invertible on $\mathrm{Lie}(U^-)(R)$; and
- the same condition holds with the roles of U and U^- reversed.

The relevance of the first condition will become clear, while the second and third are used via the following lemma.

Lemma 4.1. *Suppose that $a, b \in M^\circ(\overline{\mathbb{Q}})$ satisfy the condition that, if χ is the characteristic polynomial of ad_a on $\mathrm{Lie} P_{\overline{\mathbb{Q}}} = \mathrm{Lie} M_{\overline{\mathbb{Q}}} \oplus \mathrm{Lie} U_{\overline{\mathbb{Q}}}$, then $\chi(\mathrm{ad}_b)$ is invertible on $\mathrm{Lie}(U_{\overline{\mathbb{Q}}})$, and similarly with the roles of U and U^- reversed.*

If $b = gag^{-1}$ for some g in $G^\circ(\overline{\mathbb{Q}})$, then $g \in M^\circ(\overline{\mathbb{Q}})$.

Proof. Let $g \in G^\circ(\overline{\mathbb{Q}})$ such that $gag^{-1} = b$. Then ad_g takes each (generalised) eigenspace of ad_a to a generalised eigenspace of ad_b with the same eigenvalue. The assumption on a and b then implies that $\mathrm{Lie}(M_{\overline{\mathbb{Q}}}) \oplus \mathrm{Lie}(U_{\overline{\mathbb{Q}}})$ is stable under ad_g , from which it follows that the parabolic subgroup $M_{\overline{\mathbb{Q}}}^\circ U_{\overline{\mathbb{Q}}}$ is normalised by g . Since a parabolic subgroup is its own normaliser, we see that $g \in M^\circ U$. Arguing similarly with U and U^- reversed, we see that $g \in M^\circ U \cap M^\circ U^- = M^\circ$ as required. \square

Write $M^{d,a}$ for the open subscheme of $M^\circ \rtimes \mathrm{Fr}$ consisting of those Φ avoidant at depth d . Let $\mathfrak{X}_M^{d,a} = \{\rho \in \mathfrak{X}_M^d : \rho(\mathrm{Fr}) \in M^{d,a}\}$, and let $\mathfrak{X}_M^a = \bigcup_{d \geq 1} \mathfrak{X}_M^{d,a}$. We have a morphism

$$c : G^\circ \times \mathfrak{X}_M \rightarrow \mathfrak{X}_G$$

given by $c(g, \rho) = g\rho g^{-1}$. This descends to a morphism $\bar{c} : G^\circ \times^{M^\circ} \mathfrak{X}_M \rightarrow \mathfrak{X}_G$.

Theorem 4.2. (1) *The open subscheme $\mathfrak{X}_M^a \subset \mathfrak{X}_M$ is fibrewise dense.*

(2) *The restriction of c to $G^\circ \times \mathfrak{X}_M^a$ is smooth.*

(3) *The restriction of \bar{c} to $G^\circ \times^{M^\circ} \mathfrak{X}_M^a$ is étale.*

Proof. If $M = G$ then all of these statements are trivial. So assume that M is a proper standard Levi subgroup. It is enough to prove the theorem for $\mathfrak{X}_M^{d,a}$ for each fixed $d \geq 1$, so fix such a d .

(1) Let $J \subsetneq \Delta$ be the set of simple roots associated to P . As in the proof of [Bor79] Lemma 3.5, if we let A be the maximal subtorus of $Z(M^\circ)$ and

$$S = \{s \in A \cap G_{\text{der}}^\circ : \alpha(s) = \beta(s) \text{ for all } \alpha, \beta \in \Delta \setminus J\},$$

then S is a rank one torus and $M = Z_G(S)$. If $\lambda \in \Delta \setminus J$, then the weights of S acting adjointly on $\text{Lie}(U)$ (respectively, $\text{Lie}(U^-)$) are positive (respectively, negative) powers of $\lambda|_S$, which is independent of the choice of λ by definition. Since W_F^0 acts on $S \cong \mathbb{G}_m$ by automorphisms and also preserves U and $\lambda|_S$, it must fix S . For $s \in S$, let χ_s be the unramified character of W_F^0 sending Fr to s .

Suppose that $x \in \text{Spec } \mathbb{Z}[1/p]$ with residue field κ and that ρ is a closed point of the fibre $\mathfrak{X}_{M,\kappa}^d$. Consider the map

$$\begin{aligned} S_\kappa &\rightarrow \mathfrak{X}_{M,\kappa} \\ s &\mapsto \chi_s \rho. \end{aligned}$$

The eigenvalues of $\text{ad}_{s\rho(\text{Fr})}$ acting on $\text{Lie}(U)$ are of the form $\lambda(s)^n \mu$ for μ an eigenvalue of $\text{ad}_{\rho(\text{Fr})}$ and n positive. Similarly, for each $\gamma \in \Gamma$, the eigenvalues of $\text{ad}_{s^\gamma \rho(\text{Fr})}$ on $\text{Lie}(U^-)$ are of the form $\lambda(s)^n \mu'$ for μ' an eigenvalue of $\text{ad}_{\rho(\text{Fr})}$ and n negative. From this and the same considerations with the roles of U and U^- reversed, it follows that $s\rho(\text{Fr})$ is avoidant at depth d for all but finitely many $s \in S_\kappa$. In particular, ρ is in the closure of $\mathfrak{X}_{M,\kappa}^{d,a}$ as required.

(2) Let Z be the locally closed subscheme of \mathfrak{X}_G^d given by

$$Z = \{\rho \in \mathfrak{X}_G^d : \rho(\text{Fr}) \in M^{d,a}\}.$$

(a) We first show that the morphism $G^\circ \times Z \rightarrow \mathfrak{X}_G^d$ sending (γ, ρ) to $\gamma\rho\gamma^{-1}$ is smooth. This map is the pullback of the conjugation map $c : G^\circ \times M^{d,a} \rightarrow G$ along the projection $\mathfrak{X}_G^d \rightarrow G$ sending ρ to $\rho(\text{Fr})$. It is enough to show that c is smooth. Since $G^\circ \times M^{d,a}$ and G are smooth over $\mathbb{Z}[1/p]$, it is enough (by the fibrewise criterion of smoothness, [Gro67, Proposition 17.8.2], together with [Gro67, Proposition 17.7.1 and Théorème 17.11.1(d)]) to show that the derivative Dc is surjective on geometric tangent spaces. Since the map c is equivariant for the action of G° we may check this at a point $(e, m) \in (G^\circ \times M^{d,a})(\bar{\kappa})$ for an algebraically closed field κ . Identify the tangent spaces of $G_\kappa^\circ \times M_\kappa^{d,a}$ (resp. G_κ) at (e, m) (resp. m) with $\text{Lie } G_\kappa \oplus \text{Lie } M_\kappa$ (resp. $\text{Lie } G_\kappa$) via left multiplication by (e, m) (resp. m). A computation shows that for $(X, Y) \in \text{Lie } G_\kappa \oplus \text{Lie } M_\kappa$

$$Dc(X, Y) = (\text{ad}_m^{-1} - 1)X + Y,$$

so that Dc is surjective by the assumption that m is avoidant at depth d .

- (b) We must show that the closed immersion $\iota : \mathfrak{X}_M^{d,a} \rightarrow Z$ is an isomorphism. Since \mathfrak{X}_G is $\mathbb{Z}[1/p]$ -flat and reduced, Z also has these properties. Since ι is a closed immersion of $\mathbb{Z}[1/p]$ -flat schemes, it is enough to show that $\iota \otimes_{\mathbb{Z}[1/p]} \mathbb{Q}$ is an isomorphism.
- (c) Since $Z \otimes_{\mathbb{Z}[1/p]} \mathbb{Q}$ is reduced and of finite type over \mathbb{Q} , it is enough to show that $\iota \otimes \overline{\mathbb{Q}}$ is a bijection on closed points. Thus we must show that, if $\rho \in Z(\overline{\mathbb{Q}})$, then $\rho \in \mathfrak{X}_M^d(\overline{\mathbb{Q}})$. In other words, if ρ is an L -parameter with $\rho(\text{Fr}) \in M^{d,a}(\overline{\mathbb{Q}})$, then we also have $\rho(\sigma) \in M(\overline{\mathbb{Q}})$ and $\rho(g) \in M(\overline{\mathbb{Q}})$ for all $g \in P_F/P_F^d$. Let ρ be such a parameter, and let $\rho(\sigma) = (\Sigma_s, \sigma)\Sigma_u$ be the Jordan decomposition of $\rho(\sigma)$ (we must have that $\rho(\sigma)_u \in G^\circ(\overline{\mathbb{Q}})$ since $\overline{\mathbb{Q}}$ has characteristic zero).
- (d) We show that $\Sigma_u \in M^\circ(\overline{\mathbb{Q}})$. Write $\Sigma_u = \exp(N)$ for $N \in \text{Lie}(G)(\overline{\mathbb{Q}})$. Then (by uniqueness of Jordan decomposition) we have $\text{ad}_{\rho(\text{Fr})}(N) = qN$. By our assumption that $\rho(\text{Fr})$ is avoidant at depth d , we have $N \in \text{Lie}(M)(\overline{\mathbb{Q}})$ and so $\Sigma_u \in M^\circ(\overline{\mathbb{Q}})$.
- (e) We next show that $\Sigma_s \in M^\circ(\overline{\mathbb{Q}})$. Let r be as above. Then

$$\rho(\text{Fr})^r (\Sigma_s, \sigma) \rho(\text{Fr})^{-r} = (\Sigma_s, \sigma)^{q^r} = (\Sigma_s, \sigma),$$

which may be rewritten as

$$(\Sigma_s)^{-1} \rho(\text{Fr})^r \Sigma_s = \sigma \rho(\text{Fr})^r.$$

Applying Lemma 4.1, we see that $\Sigma_s \in M^\circ(\overline{\mathbb{Q}})$ as required.

- (f) Finally, we show that $\rho(g) \in M(\overline{\mathbb{Q}})$ for $g \in P_F/P_F^d$. This is identical to the previous step: if $\pi = \rho^\circ(g)$ we obtain

$$\pi^{-1} \rho(\text{Fr})^r \pi = {}^g \rho(\text{Fr})^r$$

and therefore that $\pi \in M^\circ(\overline{\mathbb{Q}})$.

- (3) It follows from part 2 that \bar{c} is smooth of relative dimension

$$\dim(G^\circ \times^{M^\circ} \mathfrak{X}_M^a) - \dim \mathfrak{X}_G = \dim G^\circ - \dim M^\circ + \dim \mathfrak{X}_M - \dim \mathfrak{X}_G = 0,$$

by Theorem 2.1, and hence étale. □

Remark 4.3. In fact, \bar{c} descends further to a map $G^\circ \times^{N_{G^\circ}(M^\circ)} \mathfrak{X}_M^a \rightarrow \mathfrak{X}_G$ which one can show is an open immersion. This implies that \bar{c} is a torsor under

$$N_{G^\circ}(M^\circ)/M^\circ \cong N_W(W_{M^\circ})/W_{M^\circ}$$

(see [MT11] Corollary 12.11 for this isomorphism).

Remark 4.4. The formation of \mathfrak{X}_M^a is compatible with base change, as is the property of fibrewise density, and so we obtain Theorem 4.2 over any $\mathbb{Z}[1/p]$ -algebra Λ .

5. (NON-)EXAMPLES

In all the following examples we consider unramified G and tamely ramified L -homomorphisms only. If κ is an algebraically closed field of characteristic distinct from p , then the irreducible components of $\mathfrak{X}_{G,\kappa}$ are in bijection with the $G^\circ(\kappa)$ -conjugacy classes of pairs (Σ, C) , where $\Sigma \in G^\circ(\kappa)$ and C is an element of

$$\pi_0(\{\Phi \in G_\kappa^\circ : \Phi^{\text{Fr}} \Sigma \Phi^{-1} = \Sigma^q\}).$$

If we fix Σ and fix Φ_0 with $\Phi_0 \Sigma \Phi_0^{-1} = \Sigma^q$, then by Corollary 2.7 the equivalence classes of pairs (Σ, C) are in bijection with the Ad_{Φ_0} -twisted conjugacy classes in $\pi_0(C_{G^\circ}(\Sigma))$. We call the corresponding components the components of type Σ .

Let $\Lambda = W(\overline{\mathbb{F}}_l)$; in the rest of this section, we consider everything over Λ . Our main interest is in describing the local geometry of \mathfrak{X}_G around generic points of components of the special fibre. By [DHKM20] section 4.2, it is possible to reduce the general case to that of tame parameters such that the semisimple part of Σ has l -power order — and so, modulo l , Σ is unipotent. We therefore focus on such components in the discussion below. We call the irreducible components of $\mathfrak{X}_{G, \overline{\mathbb{F}}_l}^{\text{tame}}$ on which Σ is unipotent the *unipotent components*. We have the morphism $\text{Ch} : \mathfrak{X}_G^{\text{tame}} \rightarrow (T//W)^{\text{Fr}^{-1}[q]}$ and the unipotent components are those in the preimage of the identity $\bar{e} \in (T//W)^{\text{Fr}^{-1}[q]}(\overline{\mathbb{F}}_l)$.

5.1. GL_n . Since centralizers are connected in $G = GL_n$, the irreducible unipotent components of $\mathfrak{X}_{G, \overline{\mathbb{F}}_l}$ correspond to unipotent conjugacy classes in GL_n , which are parametrized by partitions of n giving the Jordan canonical form of Σ . For every conjugacy class $[\Sigma]$, we can find a representative such that Σ is regular in a standard Levi subgroup M (the one corresponding to the same partition of n). By Theorems 4.2 and 3.2, we have open subsets $U \subset \mathfrak{X}_M$ and $V \subset \mathfrak{X}_G$, each containing the generic point of the component corresponding to Σ , and smooth morphisms

$$G \times U \rightarrow V$$

and

$$U \rightarrow (T//W_M)^{[q]}.$$

In particular, the completed local ring of \mathfrak{X}_G at a point in $V(\overline{\mathbb{F}}_l)$ will be formally smooth over $(T//W_M)^{[q]}_{\bar{e}}$. We therefore recover the geometric results of [Sho22] in this case (and there is no need to assume that Σ is unipotent here).

5.2. SL_2 . Suppose that $G = SL_2$ (with Γ trivial). There are two conjugacy classes of unipotent Σ .

If $\Sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then we may take $\Phi_0 = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. We have $\pi_0(C_{G^\circ}(\Sigma)) = \pi_0(Z(SL_2)) = \pi_0(\mu_2)$, and so there are two components of $\mathfrak{X}_{\overline{\mathbb{F}}_l}$ of type Σ unless $l = 2$, in which case there is only one. If $l \neq 2$, then there is an open subset U of \mathfrak{X}_G intersecting each of these components such that the map

$$U \rightarrow (T//W)^{[q]} = \text{Spec } \mathbb{Z}_l[x + x^{-1}] / ((x^q + x^{-q} - (x + x^{-1})))$$

is smooth.

If $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then there is only one component C_{nr} of type Σ . There is an open subset V of $\mathfrak{X}_T^{\text{tame}} \cong \mu_{q-1} \times \mathbb{G}_m$ such that the morphism $G \times V \rightarrow \mathfrak{X}_G$ is smooth and has image intersecting C_{nr} (necessarily in an open subset).

5.3. **Unitary groups.** We take G to be the L -group of an unramified quasisplit unitary group over F . Thus,

$$G = G^\circ \rtimes \text{Gal}(E/F)$$

where $G^\circ = GL_n$, the extension E/F is unramified quadratic, and the nontrivial element $c \in \text{Gal}(E/F)$ acts on G° as

$${}^c g = Jg^{-T}J^{-1}$$

for J an antidiagonal matrix of alternating 1s and -1 s. Then c preserves the standard maximal torus T and Borel B in G° .

Since centralizers in G° are connected, the irreducible components of $\mathfrak{X}_{G, \overline{\mathbb{F}}_l}$ are in bijection with $G^\circ(\overline{\mathbb{F}}_l)$ -conjugacy classes of $\Sigma \in G^\circ(\overline{\mathbb{F}}_l)$ such that ${}^c\Sigma \sim \Sigma^q$. In particular, the unipotent irreducible components are in bijection with the unipotent conjugacy classes in $G^\circ(\overline{\mathbb{F}}_l) \cong GL_n(\overline{\mathbb{F}}_l)$, since ${}^c\Sigma \sim \Sigma$ for Σ unipotent.

Note that Fr acts on $T//W$ in the same way as the map induced by $t \mapsto t^{-1}$. We therefore have a formally smooth morphism $\mathfrak{X}_G^{\text{reg}} \rightarrow (T//W)^{[-q]}$, by Theorem 3.2.

However, it is no longer true that every unipotent conjugacy class of G° contains an element regular in a standard Levi subgroup of G . Indeed, such Levi subgroups must be preserved by c , and so we only get partitions of n for which all except at most one part occurs an even number of times.

For example, when $n = 3$, the unipotent components of \mathfrak{X}_G to which Theorem 4.2 applies are those corresponding to the partitions $(1, 1, 1)$ and (3) ; we do not obtain a description of the local geometry of \mathfrak{X}_G about points for which Σ is unipotent of type $(2, 1)$.

5.4. GSp_4 . Let $G = GSp_4$, with $l > 2$. We take the symplectic form to be that given by the matrix

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

We describe the unipotent components of $\mathfrak{X}_{\overline{\mathbb{F}}_l}$. The conjugacy class of a unipotent matrix $u \in GSp_4(\overline{\mathbb{F}}_l)$ is determined by the rank of $u - 1$. The centralizer of u is connected unless the rank of $u - 1$ is two, in which case the centralizer has two connected components (see the table on p400 of [Car85]). We label the unipotent components by the rank of $\Sigma - I$ at a generic point, so that they are C_0 , C_1 , C_{2A} , C_{2B} , and C_3 .

Firstly, on the unramified component C_0 we have $\Sigma = I$. We may apply Theorem 4.2 to this component, with the Levi subgroup being the standard maximal torus, to see that there is an étale neighbourhood U of the generic point of C_0 such that U is smooth over $T^{[q]} \cong \mu_{q-1}^3$.

Secondly, we have the component C_1 on which $\Sigma - I$ has rank 1 (generically). There is a unique such component since the centralizer of any Σ such that $\Sigma - I$ has rank 1 is connected. Letting $M \cong GL_2 \times \mathbb{G}_m$ be the Levi subgroup of block diagonal matrices with block sizes 1, 2, 1, there is an étale neighbourhood U of the generic point of C_1 such that U is smooth over $(T//W_M)^{[q]}$.

Thirdly, we have the component C_3 on which $\Sigma - I$ has rank 3 (generically). There is an open neighbourhood U of the generic point of C_3 that is smooth over $(T//W)^{[q]}$.

Lastly, consider

$$u = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $C_G(u)$ has two components. The Frobenius-twisted conjugacy classes of $\pi_0(C_G(u)) \cong \mathbb{Z}/2$ are in this case just the conjugacy classes and there are therefore

two components of \mathfrak{X}_u . One of them, which we call C_{2A} , contains the points with $\Sigma = u$ and

$$\rho(\text{Fr}) = \begin{pmatrix} \lambda q & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

while the other, which we call C_{2B} , contains the points with $\Sigma = u$ and

$$\rho(\text{Fr}) = \begin{pmatrix} 0 & 0 & -\lambda q & 0 \\ 0 & 0 & 0 & \lambda \\ -q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The component C_{2A} may be understood by applying Theorem 4.2 with the Levi subgroup $M \cong GL_2 \times_{\mathbb{G}_m} GL_2$ consisting of block diagonal matrices with block sizes $(2, 2)$, and we see that an étale neighbourhood U of the generic point of C_{2A} will be smooth over $(T//W_M)_1^{[q]}$. The component C_{2B} , however, falls outside the scope of our results.

Remark 5.1. We can show by explicit calculation that, if $q \equiv -1 \pmod{l}$, then any open neighbourhood of the generic point of C_{2B} intersects the components of the generic fibre of type C_{2B} and C_3 , so that no statement like 4.2 can be true.

If $q \equiv 1 \pmod{l}$ and $l \geq 5$, then one can actually prove a statement like Theorem 4.2 for the component C_{2B} with M taken to be the pseudo-Levi subgroup (in the sense of [MS03])

$$M = Z_{\text{GSp}_4}(\Phi_0)$$

where

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

For this, we choose $\lambda = 1$ and make a change of basis such that $\rho(\text{Fr}) = \Phi_0$ and

$$\Sigma = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M.$$

In this case, we can show that an étale neighbourhood of C_{2B} is smooth over $(T//W_M)_1^{[q]}$. Is there a more general picture involving pseudo-Levi subgroups when $q \equiv 1 \pmod{l}$ (and l is sufficiently large)?

5.5. Distinguished unipotent elements. The Bala–Carter theorem (see [Pre03]) says that, in good characteristic, every unipotent element of a reductive group is a *distinguished* unipotent element of a Levi subgroup. For $G = GL_n$, the notions of distinguished and regular coincide, but in general they do not.

Concretely, suppose that $G = \mathrm{GSp}_6$, $l > 2$, and that

$$\Sigma = \begin{pmatrix} 1 & 1 & & & & \\ & 1 & & & 1 & \\ & & 1 & 1 & & \\ & & & 1 & & \\ & & & & 1 & -1 \\ & & & & & 1 \end{pmatrix}.$$

Then there is a single component of type Σ . This Σ is distinguished unipotent, so is contained in no proper Levi subgroup of G , but is not regular.⁶ This component therefore falls outside the scope of our results, and we do not understand the deformation rings at generic points of these components (except for *minimal* deformations as in [Boo19]).

Remark 5.2. If $q \equiv 1 \pmod{l}$ then the trivial representation lies on the component of type Σ . In particular, to apply the Ihara avoidance method of [Tay08] to GSp_{2n} , $n > 2$, one would need to understand something about deformations of representations of type Σ . For GSp_4 , where Ihara avoidance was worked out in section 7.4.5 of [BCGP21], the issue does not arise, as C_{2B} does not contain the trivial representation.

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⁶Indeed, it has Jordan blocks of shape $(4, 2)$ but no proper Levi subgroup of G contains elements of this shape.

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