LOCAL DEFORMATION RINGS FOR GL_2 AND A BREUIL–MÉZARD CONJECTURE WHEN $l \neq p$

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Abstract. We compute the deformation rings of two dimensional mod l representations of $\mathrm{Gal}(\overline{F}/F)$ with fixed inertial type, for l an odd prime, p a prime distinct from l, and $F/\mathbb{Q}_p$ a finite extension. We show that in this setting an analogue of the Breuil–Mézard conjecture holds, relating the special fibres of these deformation rings to the mod l reduction of certain irreducible representations of $GL_2(\mathcal{O}_F)$.

1. Introduction

Let p be a prime, and let F be a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_F. We study the (framed) deformation rings for two-dimensional mod l representations of G_F, where l is an odd prime distinct from p. More specifically, let E be a finite extension of $\mathbb{Q}_l$, with ring of integers $\mathcal{O}$, uniformiser λ, and residue field $\mathbb{F}$. Let

$$\overline{\rho} : G_F \to GL_2(\mathbb{F})$$

be a continuous representation. Then there is a universal lifting (or framed deformation) ring $R^{\square}(\overline{\rho})$ parametrising lifts of $\overline{\rho}$. Our main result relates congruences between irreducible components of $\operatorname{Spec} R^{\square}(\overline{\rho})$ to congruences between certain representations of $GL_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of F. Our method is to give explicit equations for the components of $\operatorname{Spec} R^{\square}(\overline{\rho})$, which may be of independent use.

If $\tau : I_F \to GL_2(E)$ is a continuous representation that extends to a representation of G_F (an *inertial type*), then we say that a representation $\rho : G_F \to GL_2(\overline{E})$ has type τ if its restriction to I_F is isomorphic to τ. Say that an irreducible component of $\operatorname{Spec} R^{\square}(\overline{\rho})$ has type τ if a Zariski dense subset of its $\overline{E}$-points correspond to representations of type τ. We define (definition 4.1) a formal sum $\mathcal{C}(\overline{\rho}, \tau)$ of irreducible components of the special fibre $\operatorname{Spec} R^{\square}(\overline{\rho}) \otimes_{\mathcal{O}} \mathbb{F}$. For semisimple τ, this is obtained as the intersection with the special fibre of those components of $\operatorname{Spec} R^{\square}(\overline{\rho})$ having type τ; for non-semisimple τ this must be slightly modified.

To an inertial type τ we also associate an irreducible E-representation $\sigma(\tau)$ of $GL_2(\mathcal{O}_F)$, by a slight variant on the definition of [Hen02] (see section 3.3). For an irreducible $\mathbb{F}$-representation θ of $GL_2(\mathcal{O}_F)$, define $m(\theta, \overline{\sigma(\tau)})$ to be the multiplicity of θ as a Jordan–Hölder factor of the mod λ reduction of $\sigma(\tau)$. Then we can state our main theorem (theorem 4.2):

Theorem. *Let $\overline{\rho} : G_F \to GL_2(\mathbb{F})$ be a continuous representation. For each irreducible $\mathbb{F}$-representation θ of $GL_2(\mathcal{O}_F)$, there is a formal sum $\mathcal{C}(\overline{\rho}, \theta)$ of irreducible components of $\operatorname{Spec} R^{\square}(\overline{\rho}) \otimes \mathbb{F}$ such that, for each inertial type τ, we have the* *equality*

$$\mathcal{C}(\overline{\rho},\tau)=\sum_{\theta} m(\theta,\overline{\sigma(\tau)})\mathcal{C}(\overline{\rho},\theta).$$

In fact the $\mathcal{C}(\overline{\rho},\theta)$ are uniquely determined (at least for those θ which actually occur in some $\overline{\sigma(\tau)}$).

This theorem is an analogue for mod l representations of G_F of the Breuil–Mézard conjecture [BM02], which pertains to mod p representations of $G_{\mathbb{Q}_p}$. Our statement is not in the language of Hilbert–Samuel multiplicities used in [BM02], but rather in the geometric language of [EG14]. The original conjecture of Breuil–Mézard was proved in most cases by Kisin [Kis09a]; further cases were proved by Paškūnas [Paš15] by local methods, and the full conjecture was proved when $p>3$ in [HT13]. The conjecture was generalised to n-dimensional representations of G_F in [EG14]; the only case known, outside of those just mentioned, is that of two-dimensional potentially Barsotti–Tate representations (see [GK14]).

In the $l\neq p$ setting, a comparison of special fibres of (very particular) local deformation rings was used by Taylor in [Tay08] to prove the change of level results needed to obtain non-minimal automorphy lifting theorems; this is another motivation for our result.

Our method of proof is to completely explicitly determine equations for deformation rings of fixed type, and indeed obtaining these explicit descriptions is another goal of this paper. We reduce to the tamely ramified case, in which we use the relation

$$\phi\sigma\phi^{-1}=\sigma^q$$

for $\phi\in G_F$ a lift of Frobenius and $\sigma\in I_F$ a generator of tame inertia. Since we are considering lifts ρ of fixed type, and so with fixed characteristic polynomial of $\rho(\sigma)$, we may use the Cayley–Hamilton theorem to reduce this equation to one of degree at most two in the entries of $\rho(\phi)$ and $\rho(\sigma)$. These explicit descriptions show that the irreducible components of Spec $R^{\square}(\overline{\rho})\otimes\overline{E}$ are always smooth (which is also proved in [Pil08]), and that the reduced deformation rings in which the semisimplification of the restriction to inertia is fixed are always Cohen–Macaulay (see 5.5). It is natural to ask whether these properties persist beyond the case of two dimensional representations. We note that the generic fibres of our local deformation rings have been studied in [Pil08] and [Red], but their methods say little about the integral structure.

In a forthcoming paper, we will extend theorem 4.2 to the case of n-dimensional representations using global methods.

The structure of this paper is as follows. In section 2 we define the universal deformation rings and show how to reduce their study to the case when $\overline{\rho}$ is tamely ramified. We also prove some lemmas that will be useful in the calculations that follow. In section 3 we define the deformation rings with fixed inertial type that we will need, and discuss the construction of the representations $\sigma(\tau)$. In section 4 we state and prove the main theorem, modulo the calculations of section 5 and results of section 6. Section 5 contains the calculations of explicit equations for local deformation rings, divided into cases according to the value of q mod l. Finally, in section 6 we prove the results on the mod l reduction of the $\sigma(\tau)$ that are stated in section 3.4 (and used in the proof of theorem 4.2).

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2. Preliminaries

2.1. **Fields and Galois groups.** Suppose that $l \neq p$ are primes with $l > 2$.

Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p}_F$, uniformiser ϖ_F and residue field k_F of order q. Let F have absolute Galois group G_F, inertia group I_F, and wild inertia group P_F. Let $I_F \twoheadrightarrow I_F/\tilde{P}_F \cong \mathbb{Z}_l$ be the maximal pro-l quotient of I_F, so that $\tilde{P}_F/P_F \cong \prod_{l' \neq l,p} \mathbb{Z}_{l'}$. Note that $\tilde{P}_F$ is normal in G_F and write $T_F = G_F/\tilde{P}_F$. The short exact sequence $1 \to I_F/\tilde{P}_F \to T_F \to G_F/I_F \to 1$ splits, so that $T_F \cong \mathbb{Z}_l \rtimes \hat{\mathbb{Z}}$. We fix topological generators σ of this $\mathbb{Z}_l$ and ϕ of this $\hat{\mathbb{Z}}$ such that ϕ is a lift of arithmetic Frobenius. Then the action of $\hat{\mathbb{Z}}$ on $\mathbb{Z}_l$ is given by

$$\phi\sigma\phi^{-1} = \sigma^q. \tag{1}$$

Let L/F be an unramified quadratic extension, with residue field k_L.

Now let $E/\mathbb{Q}_l$ be a finite extension with ring of integers $\mathcal{O}$, residue field $\mathbb{F}$ and uniformiser λ. Let $\epsilon : G_F \to \mathbb{Z}_l^\times$ be the l-adic cyclotomic character, and let $\mathbb{1} : G_F \to \mathbb{Z}_l^\times$ be the trivial character. If A is any $\mathcal{O}$-algebra then we will regard these as maps to $A^\times$ via the structure maps $\mathbb{Z}_l \to \mathcal{O} \to A$.

Define two integers a and b by $a = v_l(q-1)$ and $b = v_l(q+1)$, where v_l is the l-adic valuation; at most one of a and b is non-zero, since l is odd.

2.2. **Deformation rings.** Suppose that $\overline{M}$ is an n-dimensional $\mathbb{F}$-vector space and that $\overline{\rho} : G_F \to GL(\overline{M})$ is a continuous representation. Let $(\overline{e}_i)_{i=1}^n$ be a basis for $\overline{M}$, so that $\overline{\rho}$ gives a map $\overline{\rho} : G_F \to GL_n(\mathbb{F})$.

Let $\mathcal{C}_\mathcal{O}$ denote the category of artinian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$, and $\mathcal{C}_\mathcal{O}^\wedge$ the category of complete noetherian local $\mathcal{O}$-algebras with residue field $\mathbb{F}$. If A is an object of $\mathcal{C}_\mathcal{O}$ or $\mathcal{C}_\mathcal{O}^\wedge$, let $\mathfrak{m}_A$ be its maximal ideal. Define two functors

$$D(\overline{\rho}), D^\square(\overline{\rho}) : \mathcal{C}_\mathcal{O} \to \mathbf{Set}$$

as follows:

- $D(\overline{\rho})(A)$ is the set of equivalence classes of (M, ι) where: M is a free rank n A-module, $\rho : G_F \to \mathrm{Aut}_A(M)$ is a continuous homomorphism, and $\iota : M \otimes_A \mathbb{F} \xrightarrow{\sim} \overline{M}$ is an isomorphism commuting with the actions of G_F;
- $D^\square(\overline{\rho})(A)$ is the set of equivalence classes of $(M, \rho, (e_i)_{i=1}^n)$ where: M is a free rank n A-module, $\rho : G_F \to \mathrm{Aut}_A(M)$ is a continuous homomorphism and $(e_i)_{i=1}^n$ is a basis of M as an A-module, such that the isomorphism

$\iota : M \otimes_A \mathbb{F} \xrightarrow{\sim} \overline{M}$ defined by $\iota : e_i \otimes 1 \mapsto \overline{e}_i$ commutes with the actions of G_F.

In the first case, (M, ρ, ι) and (M', ρ', ι') are equivalent if there is an isomorphism $\alpha : M \to M'$, commuting with the actions of G_F, such that $\iota = \iota' \circ \alpha$; in the second case, $(M, \rho, (e_i)_i)$ and $(M', \rho', (e'_i)_i)$ are isomorphic if the map $M \to M'$ defined by $e_i \mapsto e'_i$ commutes with the actions of G_F. There is a natural transformation of functors $D^\square(\overline{\rho}) \to D(\overline{\rho})$ given by forgetting the basis.

Alternatively, when $\overline{\rho}$ is regarded as a homomorphism to $GL_n(\mathbb{F})$, we have the equivalent definitions

$$D^\square(\overline{\rho})(A) = \{\text{continuous } \rho : G_F \to GL_n(A) \text{ lifting } \overline{\rho}\}$$

and

$$D(\overline{\rho})(A) = \{\text{continuous } \rho : G_F \to GL_n(A) \text{ lifting } \overline{\rho}\}/\text{conjugacy by } 1 + M_n(\mathfrak{m}_A).$$

The functor $D(\overline{\rho})$ is not usually pro-representable, but the functor $D^\square(\overline{\rho})$ always is (see, for example, [Kis09b] (2.3.4)):

Definition 2.1. The *universal lifting ring* (or universal framed deformation ring) of $\overline{\rho}$ is the object $R^\square(\overline{\rho})$ of $\mathcal{C}^\wedge_\mathcal{O}$ that pro-represents the functor $D^\square(\overline{\rho})$. The universal lift is denoted $\rho^\square : G_F \to GL_n(R^\square(\overline{\rho}))$.

Recall the following calculation (see e.g. [BLGGT14] section 1.2):

Lemma 2.2. *The ring $R^\square(\overline{\rho})[1/l]$ is generically formally smooth of dimension n^2.*

The next lemma enables us to reduce to the case where the residual representation is trivial on $\tilde{P}_F$. Suppose that θ is an irreducible $\mathbb{F}$-representation of $\tilde{P}_F$. Then by [CHT08], lemma 2.4.11, there is a lift of θ to an $\mathcal{O}$-representation of $\tilde{P}_F$, which may be extended to an $\mathcal{O}$-representation $\tilde{\theta}$ of G_θ, where G_θ is the group $\{g \in G_F : g\theta g^{-1} \cong \theta\}$. For each irreducible representation θ of $\tilde{P}_F$, we pick such a $\tilde{\theta}$ and a finite free $\mathcal{O}$-module $N(\theta)$ on which $\tilde{P}_F$ acts as $\tilde{\theta}$. If M is a set-finite $\mathcal{O}$-module with a continuous action ρ of G_F, then define

$$M_\theta = \mathrm{Hom}_{\tilde{P}_F}(\tilde{\theta}, M).$$

The module M_θ has a natural continuous action ρ_θ of G_θ given by $(gf)(v) = gf(g^{-1}v)$; the subgroup $\tilde{P}_F$ of G_θ acts trivially.

Lemma 2.3. *(Tame reduction)*

(1) *Let M be a set-finite $\mathcal{O}$-module with a continuous action of G_F. Then there is a* natural *isomorphism*

$$M = \bigoplus_{[\theta]} \mathrm{Ind}_{G_\theta}^{G_F} \left(N(\theta) \otimes_\mathcal{O} M_\theta\right),$$

where $[\theta]$ runs over G_F-conjugacy classes of irreducible representations of $\tilde{P}_F$.

(2) *The isomorphism of part (1) induces a natural isomorphism of functors:*

$$D(\overline{\rho}) \xrightarrow{\sim} \prod_{[\theta]} D(\overline{\rho}_\theta)$$

where θ runs through a set of representatives for the G_F-conjugacy classes of irreducible representations of $\tilde{P}_F$.

(3) *If* $R^\square(\overline{\rho}_\theta)$ *is the universal framed deformation ring for the representation* $\overline{\rho}_\theta$ *of* $G_\theta/\tilde{P}_F$*, then*
$$R^\square(\overline{\rho}) \cong \left(\widehat{\bigotimes}_{[\theta]} R^\square(\overline{\rho}_\theta)\right)[[X_1, \dots, X_{n^2-\sum n_\theta^2}]]$$
where $n_\theta = \dim \rho_\theta$*. This isomorphism lies above the isomorphism* $D(\overline{\rho}) \xrightarrow{\sim} \prod_{[\theta]} D(\overline{\rho}_\theta)$ *of part (2).*

Proof. The first two parts are in [CHT08]: part (1) is lemma 2.4.12 and part (2) is corollary 2.4.13. Part (3) is the refinement to framed deformations obtained by keeping track of a basis in the construction of part (1) of the proposition, as in [Cho09], proposition 2.0.5.

As [Cho09] is not easily available, we sketch the argument for part (3): let $[\theta_1], [\theta_2], \dots$ be the G_F-conjugacy classes of irreducible $\tilde{P}_F$-representations. Pick left coset representatives $(g_{ij})_j$ for G_{θ_i} in G_F. Write N_i for $N(\theta_i)$, and choose an $\mathcal{O}$-basis $(f_{ik})_k$ of N_i.

Let A be an object of $\mathcal{C}_\mathcal{O}$, M be a free rank n A-module with a continuous action of G_F, and M_{θ_i} be as above. Given (for each i) a basis $(e_{il})_{l=1}^{n_{\theta_i}}$ of M_{θ_i}, we can produce a basis $(e_{ijkl})_{j,k,l}$ of
$$M_{\theta_i} = A[G_F] \otimes_{A[G_\theta]} (N_i \otimes_\mathcal{O} M_{\theta_i})$$
defined by
$$e_{ijkl} = g_{ij} \otimes f_{ik} \otimes e_{il}.$$
Then $(e_{ijkl})_{i,j,k,l}$ is a basis of M.

Let $\mathcal{F}(A)$ be the set of $\mathbf{Y} = (Y_{ijkl,i'j'k'l'})$ which are $n \times n$ matrices of elements of $\mathfrak{m}_A$ such that
$$Y_{ijkl,i'j'k'l'} = 0 \text{ if } i = i' \text{ and } j = j' = k = k' = 1$$
(so that $n^2 - \sum n_{\theta_i}^2$ 'free' entries of $\mathbf{Y}$ remain). Then $\mathcal{F}$ defines a functor on $\mathcal{C}_\mathcal{O}$ pro-represented by $\mathcal{O}[[X_1, \dots, X_{n^2-\sum n_\theta^2}]]$ (the variables X being simply an enumeration of those $Y_{ijkl,i'j'k'l'}$ which can be non-zero).

We then have a natural transformation of functors
$$\mathcal{F} \times \prod_{[\theta]} D^\square(\overline{\rho}_\theta) \to D^\square(\overline{\rho})$$
taking the tuple $(\mathbf{Y}, (M_{\theta_i}, \rho_{\theta_i}, e_{il})_i)$ to the tuple
$$\left(\bigoplus_i \mathrm{Ind}_{G_{\theta_i}}^{G_F}(N_i \otimes_\mathcal{O} M_{\theta_i}), \bigoplus_i \mathrm{Ind}_{G_{\theta_i}}^{G_F}(\tilde{\theta}_i \otimes_\mathcal{O} \rho_{\theta_i}), (I_n + \mathbf{Y})(e_{ijkl})_{i,j,k,l}\right).$$
Then one can check (and this is what is done in [Cho09], proposition 2.0.5) that this is in fact an isomorphism, and so we get the claimed isomorphism of pro-representing objects. □

2.3. **Twisting.**

Lemma 2.4. *Suppose that* $\chi : G_F \to \mathcal{O}^\times$ *is any character. Then there is a natural isomorphism*
$$R^\square(\overline{\rho}) \xrightarrow{\sim} R^\square(\overline{\rho} \otimes \overline{\chi}).$$
Moreover, if χ_1 *and* χ_2 *satisfy* $\overline{\chi}_1 = \overline{\chi}_2$ *then they induce the same maps* $R^\square(\overline{\rho}) \otimes \mathbb{F} \xrightarrow{\sim} R^\square(\overline{\rho} \otimes \overline{\chi}_i) \otimes \mathbb{F}$.

Proof. This follows easily from the isomorphism of functors

$$D^{\square}(\overline{\rho}) \to D^{\square}(\overline{\rho} \otimes \overline{\chi})$$

given by tensoring with χ (remembering that we are considering $\mathcal{O}$-algebras). For the last statement, observe that if the functors are restricted to $\mathbb{F}$-algebras then the isomorphism only depends on $\overline{\chi}$. $\square$

Since every $\mathbb{F}$-valued character lifts to $\mathcal{O}$ (using the Teichmüller lift) this shows that $R^{\square}(\overline{\rho}) \cong R^{\square}(\overline{\rho} \otimes \overline{\chi})$ for every $\overline{\chi} : G_F \to \mathbb{F}^{\times}$.

We also need the calculation of the universal deformation ring of a character, to which some of our calculations reduce. This is completely standard, but we include it as a simple illustration of the method.

Lemma 2.5. *Let $\overline{\chi} : G_F \to \mathbb{F}^{\times}$ be a continuous character. Then*

$$R^{\square}(\overline{\chi}) = \frac{\mathcal{O}[[X, Y]]}{((1+X)^{l^a} - 1)}$$

has l^a irreducible components, indexed by the l^ath roots of unity. They are formally smooth of relative dimension one over $\mathcal{O}$.

Proof. By lemma 2.4, we may take $\overline{\chi}$ to be trivial. If χ is any lift of $\overline{\chi}$ to an object A of $\mathcal{C}_{\mathcal{O}}$, then for $g \in \tilde{P}_F$ we must have $\chi(g)^n = 1$ for some n coprime to l, and therefore $\chi(g) = 1$, so that we are reduced to considering characters of T_F. We must have that $\chi(\sigma)^q = \chi(\sigma)$ and $\chi(\sigma) \equiv 1 \mod \mathfrak{m}_A$, and therefore that $\chi(\sigma)^{l^a} = 1$. We are then free to choose $\chi(\phi)$. Writing $\chi(\sigma) = 1 + X$ and $\chi(\phi) = 1 + Y$, we have shown that

$$D^{\square}(\overline{\chi})(A) = \operatorname{Hom}_{\mathcal{C}^{\wedge}_{\mathcal{O}}}\left(\frac{\mathcal{O}[[X, Y]]}{((1+X)^{l^a} - 1)}, A\right)$$

functorially, and so the universal framed deformation ring is as claimed. $\square$

2.4. **Multiplicities and cycles.** Suppose that X is a noetherian scheme and that $\mathcal{F}$ is a coherent sheaf on X. Let Y be the scheme-theoretic support of $\mathcal{F}$, and let $d \geq \dim Y$. Let $\mathcal{Z}^d(X)$ be the free abelian group on the d-dimensional points of X; elements of $\mathcal{Z}^d(X)$ are called d-dimensional cycles. If $\mathfrak{a} \in X$ is a point of dimension d write $[\mathfrak{a}]$ for the corresponding element of $\mathcal{Z}^d(X)$ and define the multiplicity $e(\mathcal{F}, \mathfrak{a})$ to be the length of $\mathcal{F}_{\mathfrak{a}}$ as an $\mathcal{O}_{Y,\mathfrak{a}}$-module (this is zero if $\mathfrak{a} \notin Y$).

Definition 2.6. The cycle $Z^d(\mathcal{F})$ associated to $\mathcal{F}$ is the element

$$\sum_{\mathfrak{a}} e(\mathcal{F}, \mathfrak{a})[\mathfrak{a}] \in \mathcal{Z}^d(X).$$

If $X = \operatorname{Spec} A$ is affine and $\mathcal{F} = \widetilde{M}$ for a finitely generated A-module M, then we will write $Z^d(M)$ for $Z^d(\mathcal{F})$.

If $i : X \to X'$ is a closed immersion of X in a noetherian scheme X', then there is a natural inclusion $i_* : \mathcal{Z}^d(X) \to \mathcal{Z}^d(X')$ for each d. For a coherent sheaf $\mathcal{F}$ on X whose support has dimension at most d, we then have

$$i_*(Z^d(\mathcal{F})) = Z^d(i_*(\mathcal{F})).$$

We will often use this compatibility without comment.

A cycle is **effective** if it is of the form $\sum n_{\mathfrak{a}}[\mathfrak{a}]$ for $n_{\mathfrak{a}} \geq 0$. Say that an effective cycle C_1 is a **subcycle** of an effective cycle C_2 if $C_2 - C_1$ is also effective.

2.5. **A determinantal ring.** For a, b and c natural numbers, if I is the ideal generated by the $a \times a$ minors of a $b \times c$ matrix with independent indeterminant entries over a Cohen–Macaulay ring A, then A/I is always Cohen–Macaulay (see [Eis95] theorem 18.18). We include a simple proof in the very special case that we need below.

Remark. The proof given below is incorrect, but the proposition is correct. See Section 7 for details. We thank Lue Pan for pointing this out.

Proposition 2.7. *Let $k \geq 2$ be an integer and let A be either a field or a discrete valuation ring. Let $R = A[X_1, \ldots, X_k, Y_1, \ldots, Y_k]$ and let $I \lhd R$ be the ideal generated by the 2×2 minors of:*

$$\begin{pmatrix} X_1 & X_2 & \ldots & X_k \\ Y_1 & Y_2 & \ldots & Y_k \end{pmatrix}.$$

Let $S = R/I$. Then S is a Cohen–Macaulay domain and is flat over A. It is Gorenstein if and only if $k = 2$.

The same is true if we replace S by its completion $S^\wedge$ at the 'irrelevant' ideal $(X_1, \ldots, X_k, Y_1, \ldots, Y_k)$.

Proof. Note that R and S are naturally graded A-algebras.

Suppose that A is a field. It is easy to see that $\mathrm{Proj}(S)$ is a smooth irreducible projective variety over A of dimension $k + 1$ — it is covered by the open sets $\{X_i \neq 0\}$ and $\{Y_i \neq 0\}$, each of which is isomorphic to $(\mathbb{A}^1_A \setminus \{0\}) \times \mathbb{A}^k_A$. Thus S is a domain. We may extend A so that its cardinality is at least $k+1$, and choose pairwise distinct $\alpha_1, \ldots, \alpha_k \in A^\times$.

I claim that $(X_1 - \alpha_1 Y_1, \ldots, X_k - \alpha_k Y_k, Y_1 + \ldots + Y_k)$ is a regular sequence in S. To see this, observe that $\mathrm{Proj}\,(S/(X_1 - \alpha_1 Y_1, \ldots, X_i - \alpha_i Y_i))$ is reduced (we may check this on the affine pieces) and that its irreducible components are all of the form

$$\mathrm{Proj}\left(\frac{R}{(X_j - \alpha_{i_0} Y_j)_{1 \leq j \leq k} + (X_j, Y_j)_{1 \leq j \leq i, j \neq i_0}}\right)$$

for $1 \leq i_0 \leq i$ or of the form

$$\mathrm{Proj}(S/(X_1, \ldots, X_i, Y_1, \ldots, Y_i)).$$

Now it is easy to check that $X_{i+1} - \alpha_{i+1} Y_{i+1}$ (if $i < k$) or $Y_1 + \ldots + Y_k$ (if $i = k$) is a non-zerodivisor on each of these components, and so is a non-zerodivisor on $S/(X_1 - \alpha_1 Y_1, \ldots, X_i - \alpha_i Y_i)$ as required.

Now

$$S/((X_i - \alpha_i Y_i)_i, Y_1 + \ldots + Y_k) \cong A[Y_2, \ldots, Y_k]/(Y_2, \ldots, Y_k)^2$$

is Gorenstein if and only if $k = 2$, as required.

If A is a DVR then the following easy lemma (a specialisation of [Sno11] proposition 2.2.1) gives the result.

Lemma 2.8. *If A is a DVR and S is a finitely generated A-algebra such that $S \otimes A/\mathfrak{m}_A$ and $S \otimes \mathrm{Frac}\, A$ are domains of the same dimension, then S is flat over A (that is, a uniformiser of A is a regular parameter in S).*

The final statement of the proposition follows from the facts that both localisation and completion preserve the properties of being Gorenstein, Cohen–Macaulay, or A-flat; $S^\wedge$ is a domain because its associated graded ring is S, which is a domain. □

3. Types

3.1. Inertial types.

Definition 3.1. An *inertial type* τ (of dimension n) is an equivalence class of pairs (r_τ, N_τ) such that:

- $r_\tau : I_F \to GL_n(\overline{E})$ is a representation with open kernel;
- N_τ is a nilpotent $n \times n$ matrix over $\overline{E}$;
- (r_τ, N_τ) extends to a Weil–Deligne representation of G_F.

In particular, N_τ commutes with the image of r_τ. Two such pairs are equivalent if they are conjugate by an element of $GL_n(\overline{E})$.

We say that a continuous representation $\rho : G_F \to GL_n(\overline{E})$ has inertial type τ if the restriction to inertia of the associated Weil–Deligne representation is equivalent to τ.

We define some particular two-dimensional types which will often arise. They will all be of the form (r, N) with $r|_{\tilde{P}_F}$ trivial, and are therefore determined by $r(\sigma)$ and N. Define:

- $\tau_{\zeta,s}$ by $r(\sigma) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}$ and $N = 0$, where ζ is an l^ath root of unity (s is for 'split');
- $\tau_{\zeta,ns}$ by $r(\sigma) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, where ζ is an l^ath root of unity (ns is for 'non-split');
- τ_{ζ_1,ζ_1} by $r(\sigma) = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}$ and $N = 0$ where, ζ_1 and ζ_2 are distinct l^ath roots of unity;
- τ_ξ by $r(\sigma) = \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ and $N = 0$ where, ξ is a *non-trivial* l^bth root of unity.

To see that τ_ξ *is* a type, note that if L/F is the unramified quadratic extension, then there is a character of $G_L/\tilde{P}_F$ mapping σ to ξ, which when induced to G_F gives a representation of type τ_ξ.

3.2. Deformation rings with fixed type.

Definition 3.2. Let τ be an inertial type. Then $R^\square(\overline{\rho}, \tau)$ is the maximal reduced, l-torsion free quotient of $R^\square(\overline{\rho})$ with the following property: if $x : R^\square(\overline{\rho}) \to GL_n(\overline{E})$ is a continuous homomorphism such that the associated representation $\rho_x : G_F \to GL_n(\overline{E})$ has type τ, then x factors through $R^\square(\overline{\rho}, \tau)$.

The rings $R^\square(\overline{\rho}) \otimes \mathbb{F}$ and $R^\square(\overline{\rho}, \tau) \otimes \mathbb{F}$ will occur very often, and so we denote them respectively by $\overline{R}^\square(\overline{\rho})$ and $\overline{R}^\square(\overline{\rho}, \tau)$.

From now on suppose that $n = 2$. Write $\tau = (r_\tau, N_\tau)$ and assume that E is large enough that all of the roots of the characteristic polynomial of r_τ lie in E. Let $R^\square(\overline{\rho}, \tau)^\circ$ be the maximal quotient of $R^\square(\overline{\rho})$ on which:

- if r_τ is not scalar then, for all $g \in I_F$, the characteristic polynomial of $\rho^\square(g)$ agrees with that of r_τ;
- if r_τ is scalar and $N_\tau = 0$ then, for all $g \in I_F$, $\rho^\square(g)$ is scalar and agrees with r_τ;

- if r_τ is scalar and $N_\tau \neq 0$ then, for all $g \in I_F$, the characteristic polynomial of $\rho^\square(g)$ agrees with that of r_τ. Moreover, we have

$$q(\operatorname{tr}\rho^\square(\phi))^2 = (q+1)^2 \det(\rho^\square(\phi)). \tag{2}$$

It is clear that these quotients exist and that the conditions imposed are deformation problems for $\overline{\rho}$.

Lemma 3.3. *The ring $R^\square(\overline{\rho},\tau)$ is a reduced l-torsion free quotient of $R^\square(\overline{\rho},\tau)^\circ$.*

If $N_\tau = 0$, then we have that $R^\square(\overline{\rho},\tau)$ is equal to the maximal reduced l-torsion free quotient of $R^\square(\overline{\rho},\tau)^\circ$.

Proof. The first part is clear unless r_τ is scalar and $N_\tau \neq 0$. In this case, we must show that any representation $\rho : G_F \to GL_2(\overline{E})$ of type τ satisfies equation (2). The Weil–Deligne representation (r, N) corresponding to such a ρ satisfies $r|_{I_F} = r_\tau$ and $N \neq 0$. Then $r(\phi)N = qNr(\phi)$ implies that $r(\phi)$ preserves the line $\ker N$ and the quotient $\overline{E}^2/\ker N$. If it acts as α on the former and β on the latter then we must have $\alpha = q\beta$; as α and β are the eigenvalues of $\rho(\phi)$ the equation (2) is easily verified.

The final claim follows from the simple observation that any $\overline{E}$-point of $R^\square(\overline{\rho},\tau)^\circ$ has associated Galois representation of type τ, except perhaps if r_τ is scalar and $N_\tau \neq 0$. $\square$

Remark 3.4. If R is a reduced, l-torsion free quotient of $R^\square(\overline{\rho})$ such that $R^\square(\overline{\rho},\tau)$ is a quotient of R, then $R = R^\square(\overline{\rho},\tau)$ if and only if the closed points of type τ are Zariski dense in $\operatorname{Spec} R[1/l]$. In our calculations, when this is true it will always be clear by inspection.

3.3. K**-Types.** Let $G = GL_2(F)$, $K = GL_2(\mathcal{O}_F)$, and for $N \geq 1$ let $K(N) = 1 + M_2(\mathfrak{p}_F^N)$ and $K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \in \mathfrak{p}_F^N \right\}$. Let $U_0 = \mathcal{O}_F^\times$ and for $N \geq 1$ let $U_N = 1 + \mathfrak{p}_F^N$. The *exponent* of a character χ of $\mathcal{O}_F^\times$ is the smallest $N \geq 0$ such that χ is trivial on U_N. If π is an irreducible admissible representation of $GL_m(F)$ (we only need $m = 1$ and $m = 2$) over $\overline{E}$, let $\operatorname{rec}(\pi)$ be the continuous representation of W_F over $\overline{E}$ associated to π under the local Langlands correspondence (normalised so as to be preserved by automorphisms of $\overline{E}$).

For each two-dimensional inertial type $\tau = (r_\tau, N_\tau)$, we define an irreducible representation $\sigma(\tau)$ by the following recipe:

- If $\tau = \tau_{1,s}$, then $\sigma(\tau)$ is the trivial representation of K.
- If $\tau = \tau_{1,ns}$, then $\sigma(\tau)$ is the inflation to K of the Steinberg representation St of $GL_2(k_F)$.
- If $\tau = (\mathbb{1} \oplus \operatorname{rec}(\epsilon)|_{I_F}, 0)$ for a non-trivial character ϵ of $F^\times$ of exponent N, then
$$\sigma(\tau) = \operatorname{Ind}_{K_0(N)}^K \epsilon,$$
where $\epsilon\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \epsilon(a)$.
- If $\tau = (\operatorname{rec}(\pi)|_{I_F}, 0)$ for a cuspidal representation π of $GL_2(F)$, then by [BH06], 15.5 Theorem, there is a certain subgroup $J \subset G$, containing the center of G and compact modulo center, and a representation Λ of J such that
$$\pi = \text{c-Ind}_J^G \Lambda.$$

By conjugating, we may suppose that the maximal compact subgroup J^0 of J is contained in K. We then have

$$\sigma(\tau) = \mathrm{Ind}_{J^0}^{K}(\Lambda|_{J^0}).$$

- If $\tau = \tau' \otimes \mathrm{rec}(\chi)|_{I_F}$, then $\sigma(\tau) = \sigma(\tau') \otimes (\chi|_{U_0} \circ \det)$.

This is a slightly modified version of the construction in [Hen02] — the construction there only depends on r_τ, and agrees with ours whenever r_τ is not scalar. The following is an easy consequence of [Hen02]:

Proposition 3.5. *If $\sigma(\tau)$ is contained in an irreducible admissible representation π of $GL_2(F)$ and $\mathrm{rec}(\pi) = (r, N)$ then $r|_{I_F} \cong r_\tau$ and either $N \cong N_\tau$ or $N_\tau \neq 0$ and $N = 0$.*

If π is infinite-dimensional, then the converse is true.

3.4. **Reduction of types.** Suppose that $\overline{r} : I_F \to GL_2(\overline{\mathbb{F}})$ is such that $\overline{r}$ extends to G_F.

Definition 3.6. The set $L(\overline{r})$ is the set of types τ such that there exists a representation $\rho : G_F \to GL_2(\mathcal{O}_{\overline{E}})$ of type τ satisfying

$$\overline{\rho}|_{I_F} \cong \overline{r}.$$

If $\overline{r}|_{\tilde{P}_F}$ is non-scalar then we abuse notation and also write $L(\overline{r})$ for the set of r such that $(r, 0) \in L(\overline{r})$, as in this case every element of $L(\overline{r})$ is of this form.

Lemma 3.7. *Suppose that $\overline{r}$ is trivial on $\tilde{P}_F$. Then each element of $L(\overline{r})$ is one of the types $\tau_{\zeta,s}$, $\tau_{\zeta,ns}$, τ_{ζ_1,ζ_2}, τ_ξ defined in section 3.1.*

Proof. Suppose that $\rho : G_F \to GL_2(\mathcal{O}_{\overline{E}})$ is of type τ and is such that $\overline{\rho}|_{I_F} \cong \overline{r}$. As $\overline{r}|_{\tilde{P}_F}$ is trivial, ρ must also be trivial on $\tilde{P}_F$ and its type is determined by the eigenvalues of $\rho(\sigma)$ and by a nilpotent matrix N commuting with $\rho(\sigma)$. Now, the fundamental relation $\phi\sigma\phi^{-1} = \sigma^q$ shows that the eigenvalues of $\rho(\sigma)$ are the same (but perhaps in a different order) as those of $\rho(\sigma)^q$, and this implies that they are (q^2-1)th roots of unity. Moreover, they are congruent to 1 modulo the maximal ideal of $\mathcal{O}_{\overline{E}}$, and so must in fact be either l^ath or l^bth roots of unity (recall that at most one of a and b is non-zero, since $l \neq 2$). If they are distinct l^ath roots of unity, then N must be zero and $\tau = \tau_{\zeta_1,\zeta_2}$; if they are equal l^ath roots of unity then $\tau = \tau_{\zeta,s}$ or $\tau_{\zeta,ns}$; if they are l^bth roots of unity then they must be ξ and $\xi^q = \xi^{-1}$ for an l^bth root of unity ξ. Moreover the case $\xi = 1$ has already been dealt with and so we may assume that $\xi \neq 1$, in which case $N = 0$ and $\tau = \tau_\xi$. □

Lemma 3.8.
(1) *Suppose that $\overline{r}|_{\tilde{P}_F}$ is irreducible. There is a lift r of $\overline{r}$ to $GL_2(\overline{E})$, which we fix. Then $L(\overline{r}) = \{r \otimes \chi\}_\chi$ as χ runs over the set of characters $\chi : I_F \to \overline{E}^\times$ which extend to G_F and reduce to the trivial character.*
(2) *Suppose that $\overline{r}|_{\tilde{P}_F} \cong (\overline{r}_1 \oplus \overline{r}_2)|_{\tilde{P}_F}$ where $\overline{r}_1$ and $\overline{r}_2$ are distinct characters of G_F. There are lifts r_1 and r_2 of $\overline{r}_1$ and $\overline{r}_2$ to $\overline{E}^\times$, which we fix. Then $L(\overline{r}) = \{(r_1|_{I_F} \otimes \chi_1) \oplus (r_2|_{I_F} \otimes \chi_2)\}_{\chi_1,\chi_2}$ where χ_1, χ_2 run over all pairs of characters $I_F \to \overline{E}^\times$ which extend to G_F and reduce to the trivial character.*
(3) *Suppose that $\overline{r}|_{\tilde{P}_F} \cong (\overline{r}_1 \oplus \overline{r}_1^c)|_{\tilde{P}_F}$ where $\overline{r}_1$ and $\overline{r}_1^c$ are distinct characters of G_L which are conjugate by an element of G_F (recall that L/F is the unramified quadratic extension). There is a lift r_1 of $\overline{r}_1$ to $\overline{E}^\times$. Then*

$L(\overline{r}) = \{(r_1|_{I_F} \otimes \chi) \oplus (r_1^c|_{I_F} \otimes \chi^c)\}_\chi$ as χ runs over all characters $I_F \to \overline{E}^\times$ which extend to G_L and reduce to the trivial character.

Proof. This follows from proposition 5.1 below; the ingredients in the proof of that proposition are lemma 2.3 (reduction to the tame case) and lemma 2.4 (lifting ring of a character). □

Lemma 3.9. *If $\tau = (r, 0)$ is an inertial type with $r|_{\tilde{P}_F}$ non-scalar, then $\overline{\sigma(\tau)}$ is irreducible. If τ' is any other inertial type, then $\overline{\sigma(\tau')}$ contains $\overline{\sigma(\tau)}$ if and only if $\tau' \in L(\overline{r})$ (in which case $\overline{\sigma(\tau)} \cong \overline{\sigma(\tau')}$).*

Proof. These are the results of propositions 6.4 and 6.5. □

If $\tau = (r, N)$ with $r|_{\tilde{P}_F}$ scalar, then $\overline{\sigma(\tau)}$ need not be irreducible. We give the (well-known) analysis of these $\overline{\sigma(\tau)}$ in section 6.1. For now, we just give names to the following representations of $GL_2(k_F)$ (and hence, by inflation, of K) over $\mathbb{F}$:

- the trivial representation, $\mathbb{1}$;
- the Steinberg representation, St (irreducible if $q \not\equiv -1 \mod l$);
- if $q \equiv -1 \mod l$, the cuspidal (but not supercuspidal) subrepresentation π_1 of St.

4. The 'Breuil–Mézard conjecture'

Let $\overline{\rho} : G_F \to GL_2(\mathbb{F})$ be a continuous representation, and suppose that E is sufficiently large that:

- every subrepresentation of $\overline{\rho} \otimes \overline{\mathbb{F}}$ is already defined over $\mathbb{F}$;
- E contains all of the $(q^2 - 1)$th roots of unity;
- for every $\tau \in L(\overline{\rho}|_{I_F})$, $\sigma(\tau)$ is defined over E.

We state our analogue of the Breuil–Mézard conjecture when $l \neq p$. By lemma 2.2 and the fact that $R^\square(\overline{\rho}, \tau)$ is defined to be $\mathcal{O}$-flat, we have

$$\dim \overline{R}^\square(\overline{\rho}, \tau) \leq 4.$$

Definition 4.1. We associate to each type $\tau = (r, N)$ a cycle $\mathcal{C}(\overline{\rho}, \tau) \in \mathcal{Z}^4(\overline{R}^\square(\overline{\rho}))$ as follows:

- if $N = 0$, set
$$\mathcal{C}(\overline{\rho}, \tau) = Z^4(\overline{R}^\square(\overline{\rho}, \tau));$$
- if $N \neq 0$ (in which case r must be scalar) let $\tau' = (r, 0)$ and set
$$\mathcal{C}(\overline{\rho}, \tau) = Z^4(\overline{R}^\square(\overline{\rho}, \tau)) + Z^4(\overline{R}^\square(\overline{\rho}, \tau')).$$

Then we have

Theorem 4.2. *For each irreducible $\overline{\mathbb{F}}$-representation θ of $GL_2(\mathcal{O}_F)$, there is an effective cycle $\mathcal{C}(\overline{\rho}, \theta) \in \mathcal{Z}^4(\overline{R}^\square(\overline{\rho}))$ such that, for any inertial type τ, we have an equality of cycles*

$$\mathcal{C}(\overline{\rho}, \tau) = \sum_\theta m(\theta, \overline{\sigma(\tau)}) \mathcal{C}(\overline{\rho}, \theta) \tag{3}$$

where $m(\theta, \overline{\sigma(\tau)})$ is the multiplicity of θ as a Jordan–Hölder factor of $\overline{\sigma(\tau)}$ and the sum runs over all θ.

Proof. We proceed case by case, using the results of section 3.4 and of sections 5 and 6.1 below.

Suppose that $\overline{\rho}|_{\tilde{P}_F}$ is non-scalar. Then by lemma 3.9, the representations $\overline{\sigma(\tau)}$ for $\tau \in L(\overline{\rho}|_{I_F})$ are all irreducible and isomorphic to a common irreducible representation, which we call θ_0. By corollary 5.2, $\overline{R}^{\square}(\overline{\rho})$ has a unique minimal prime, denoted $\mathfrak{a}$, which has dimension 4. So we have

$$\mathcal{Z}^4(\mathrm{Spec}(\overline{R}^{\square}(\overline{\rho}))) = \mathbb{Z} \cdot [\mathfrak{a}].$$

Define $\mathcal{C}(\overline{\rho}, \theta_0) = [\mathfrak{a}]$, and $\mathcal{C}(\overline{\rho}, \theta) = 0$ for $\theta \neq \theta_0$. By corollary 5.2,

$$\mathcal{C}(\overline{\rho}, \tau) = [\mathfrak{a}] = \mathcal{C}(\overline{\rho}, \theta_0)$$

if $\tau \in L(\overline{\rho}|_{\tilde{P}_F})$ and

$$\mathcal{C}(\overline{\rho}, \tau) = 0$$

otherwise. In other words, for all τ we have

$$\mathcal{C}(\overline{\rho}, \tau) = \sum_{\theta} m(\theta, \overline{\sigma(\tau)})\mathcal{C}(\overline{\rho}, \theta)$$

as required.

If $\overline{\rho}|_{\tilde{P}_F}$ is scalar, then we may twist $\overline{\rho}$ by a character of G_F and apply lemma 2.4 and so *suppose for the rest of the proof that* $\overline{\rho}|_{\tilde{P}_F}$ *is trivial.*

If $q \neq \pm 1 \bmod l$, then $L(\overline{\rho}|_{I_F}) \subset \{\tau_{1,s}, \tau_{1,ns}\}$. By the discussion of section 6.1, we have that

$$\overline{\sigma(\tau_{1,s})} = \mathbb{1}$$

and

$$\overline{\sigma(\tau_{1,ns})} = \mathrm{St}$$

are irreducible and non-isomorphic, and that neither is a Jordan–Hölder factor of any other $\overline{\sigma(\tau)}$. So the fact that we *can* define the $\mathcal{C}(\overline{\rho}, \theta)$ so as to satisfy equation (3) is a triviality, as there are no relations amongst the $\overline{\sigma(\tau)}$ for different τ. We work out what the $\mathcal{C}(\overline{\rho}, \theta)$ are explicitly: for $\theta \neq \mathbb{1}$ or St we define $\mathcal{C}(\overline{\rho}, \theta) = 0$. Otherwise, there are four cases to consider:

- if $\overline{\rho}(\phi)$ has eigenvalues with ratio not in $\{1, \pm q\}$ then by proposition 5.3 there is a unique minimal prime $\mathfrak{a}_{nr}$ of $\overline{R}^{\square}(\overline{\rho})$. In this case, define
$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = [\mathfrak{a}_{nr}]$$
$$\mathcal{C}(\overline{\rho}, \mathrm{St}) = [\mathfrak{a}_{nr}];$$
- if $\overline{\rho}$ is an extension of the trivial character by itself then by proposition 5.5 part 1 there is a unique minimal prime $\mathfrak{a}_{nr}$ of $\overline{R}^{\square}(\overline{\rho})$. In this case, define
$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = [\mathfrak{a}_{nr}]$$
$$\mathcal{C}(\overline{\rho}, \mathrm{St}) = [\mathfrak{a}_{nr}];$$
- if $\overline{\rho}$ is a non-split extension of the trivial character by the cyclotomic character then by proposition 5.5 part 2 there is a unique minimal prime $\mathfrak{a}_N$ of $\overline{R}^{\square}(\overline{\rho})$. In this case, define
$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = 0$$
$$\mathcal{C}(\overline{\rho}, \mathrm{St}) = [\mathfrak{a}_N];$$

- if $\overline{\rho}$ is the direct sum of the trivial character and the cyclotomic character then by proposition 5.5 part 2 there are two minimal primes of $\overline{R}^{\square}(\overline{\rho})$, denoted there by $\mathfrak{a}_{nr}$ and $\mathfrak{a}_N$. In this case, define

$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = [\mathfrak{a}_{nr}]$$
$$\mathcal{C}(\overline{\rho}, \mathrm{St}) = [\mathfrak{a}_{nr}] + [\mathfrak{a}_N].$$

It is then easy to verify that equation (3) holds; we just do the last case. We see from proposition 5.5 part 2 that

$$\mathcal{C}(\overline{\rho}, \tau_{1,s}) = [\mathfrak{a}_{nr}] \qquad\qquad = \mathcal{C}(\overline{\rho}, \mathbb{1})$$
$$\mathcal{C}(\overline{\rho}, \tau_{1,ns}) = [\mathfrak{a}_{nr}] + [\mathfrak{a}_N] = \mathcal{C}(\overline{\rho}, \mathrm{St})$$

and $\mathcal{C}(\overline{\rho}, \tau) = 0$ for all other τ, exactly as required by equation (3).

If $q = -1 \bmod l$, then $L(\overline{\rho}|_{I_F}) \subset \bigcup_\xi \{\tau_{1,s}, \tau_{1,ns}, \tau_\xi\}$ for ξ a non-trivial l^bth root of unity. By the discussion of section 6.1, we have that

$$\overline{\sigma(\tau_{1,s})} = \mathbb{1},$$
$$\overline{\sigma(\tau_\xi)} = \pi_1,$$

and

$$\overline{\sigma(\tau_{1,ns})}^{ss} = \mathbb{1} \oplus \pi_1$$

where $\mathbb{1}$ and π_1 are irreducible and non-isomorphic, and are not Jordan–Hölder factors of any other $\overline{\sigma(\tau)}$. For $\theta \neq \mathbb{1}$ or π_1 we define $\mathcal{C}(\overline{\rho}, \theta) = 0$. Otherwise, there are four cases to consider:

- if $\overline{\rho}(\phi)$ has eigenvalues with ratio not in $\{\pm 1\}$ then by proposition 5.3 there is a unique minimal prime $\mathfrak{a}_{nr}$ of $\overline{R}^{\square}(\overline{\rho})$. In this case, define

$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = [\mathfrak{a}_{nr}]$$
$$\mathcal{C}(\overline{\rho}, \pi_1) = 0;$$

- if $\overline{\rho}$ is an extension of the trivial character by itself then by proposition 5.6 part 1 there is a unique minimal prime $\mathfrak{a}_{nr}$ of $\overline{R}^{\square}(\overline{\rho})$. In this case, define

$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = [\mathfrak{a}_{nr}]$$
$$\mathcal{C}(\overline{\rho}, \pi_1) = 0;$$

- if $\overline{\rho}$ is a non-split extension of the trivial character by the cyclotomic character then by proposition 5.6 part 2a there is a unique minimal prime, denoted $\mathfrak{a}_N$ in that proposition, of $\overline{R}^{\square}(\overline{\rho}, \tau_{1,ns})$, which we regard as a prime of $\overline{R}^{\square}(\overline{\rho})$. In this case, define

$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = 0$$
$$\mathcal{C}(\overline{\rho}, \pi_1) = [\mathfrak{a}_N];$$

- if $\overline{\rho}$ is the direct sum of the trivial character by the cyclotomic character then in proposition 5.6 part 2b three four-dimensional primes of $\overline{R}^{\square}(\overline{\rho})$ are defined, denoted there $\mathfrak{a}_{nr}$, $\mathfrak{a}_N$ and $\mathfrak{a}_{N'}$. In this case, define

$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = [\mathfrak{a}_{nr}]$$
$$\mathcal{C}(\overline{\rho}, \pi_1) = [\mathfrak{a}_N] + [\mathfrak{a}_{N'}].$$

It is then easy to verify that equation (3) holds using proposition 5.3 in the first case and proposition 5.6 parts 1, 2a, and 2b in the second, third, and fourth cases; again we just do the fourth case, which is the most complicated. Equation (3) is equivalent to the equations:

$$\begin{aligned}\mathcal{C}(\overline{\rho},\tau_{1,s}) = \mathcal{C}(\overline{\rho},\mathbb{1}) &= [\mathfrak{a}_{nr}]\\ \mathcal{C}(\overline{\rho},\tau_{1,ns}) = \mathcal{C}(\overline{\rho},\mathbb{1}) + \mathcal{C}(\overline{\rho},\pi_1) &= [\mathfrak{a}_{nr}] + [\mathfrak{a}_N] + [\mathfrak{a}_{N'}]\\ \mathcal{C}(\overline{\rho},\tau_\xi) = \mathcal{C}(\overline{\rho},\pi_1) &= [\mathfrak{a}_N] + [\mathfrak{a}_{N'}]\end{aligned}$$

and

$$\mathcal{C}(\overline{\rho},\tau) = 0$$

if $\tau \notin \bigcup_\xi \{\tau_{1,s},\tau_{1,ns},\tau_\xi\}$. But by proposition 5.6 part 2b we have:

$$\begin{aligned}\mathcal{C}(\overline{\rho},\tau_{1,s}) = Z^4(\overline{R}(\overline{\rho},\tau_{1,s})) &= [\mathfrak{a}_{nr}]\\ \mathcal{C}(\overline{\rho},\tau_{1,ns}) = Z^4(\overline{R}(\overline{\rho},\tau_{1,s})) + Z^4(\overline{R}(\overline{\rho},\tau_{1,ns})) &= [\mathfrak{a}_{nr}] + [\mathfrak{a}_N] + [\mathfrak{a}_{N'}]\\ \mathcal{C}(\overline{\rho},\tau_\xi) = Z^4(\overline{R}(\overline{\rho},\tau_\xi)) &= [\mathfrak{a}_N] + [\mathfrak{a}_{N'}]\end{aligned}$$

and

$$\mathcal{C}(\overline{\rho},\tau) = 0$$

if $\tau \notin \bigcup_\xi \{\tau_{1,s},\tau_{1,ns},\tau_\xi\}$, as required.

If $q = 1 \bmod l$, then $L(\overline{\rho}|_{I_F}) \subset \bigcup_{\zeta,\zeta_1,\zeta_2}\{\tau_{\zeta,s},\tau_{\zeta,ns},\tau_{\zeta_1,\zeta_2}\}$ for ζ, ζ_1 and ζ_2 (possibly trivial) l^ath roots of unity with $\zeta_1 \neq \zeta_2$. By the discussion of section 6.1, we have that

$$\begin{aligned}\overline{\sigma(\tau_{\zeta,s})} &= \mathbb{1},\\ \overline{\sigma(\tau_{\zeta,ns})} &= \mathrm{St},\end{aligned}$$

and

$$\overline{\sigma(\tau_{\zeta_1,\zeta_2})} = \mathbb{1} \oplus \mathrm{St}$$

where $\mathbb{1}$ and St are irreducible and non-isomorphic, and are not Jordan–Hölder factors of any other $\overline{\sigma(\tau)}$. For $\theta \neq \mathbb{1}$ or St we define $\mathcal{C}(\overline{\rho},\theta) = 0$. Otherwise, there are four cases to consider:

- if $\overline{\rho}(\phi)$ has eigenvalues with ratio not in $\{\pm 1\}$ then by proposition 5.3 there is a unique minimal prime $\mathfrak{a}_{nr}$ of $\overline{R}^\square(\overline{\rho})$. In this case, define

$$\begin{aligned}\mathcal{C}(\overline{\rho},\mathbb{1}) &= [\mathfrak{a}_{nr}]\\ \mathcal{C}(\overline{\rho},\mathrm{St}) &= [\mathfrak{a}_{nr}];\end{aligned}$$

- if $\overline{\rho}$ is a ramified extension of the trivial character by itself then by proposition 5.8 part 1 there is a unique minimal prime $\mathfrak{a}_N$ of $\overline{R}^\square(\overline{\rho},\tau_{1,ns})$ which we regard as a four-dimensional prime of $R^\square(\overline{\rho})$. In this case, define

$$\begin{aligned}\mathcal{C}(\overline{\rho},\mathbb{1}) &= 0\\ \mathcal{C}(\overline{\rho},\mathrm{St}) &= [\mathfrak{a}_N];\end{aligned}$$

- if $\overline{\rho}$ is a unramified extension of the trivial character by itself then by proposition 5.8 parts 2 and 3 there are four-dimensional primes of $\overline{R}^{\square}(\overline{\rho})$ which are denoted there by $[\mathfrak{a}_{nr}]$ and $[\mathfrak{a}_N]$. In this case, define

$$\mathcal{C}(\overline{\rho}, \mathbb{1}) = [\mathfrak{a}_{nr}]$$
$$\mathcal{C}(\overline{\rho}, \mathrm{St}) = [\mathfrak{a}_{nr}] + [\mathfrak{a}_N].$$

It is then easy to verify that equation (3) holds using proposition 5.3 in the first case, proposition 5.8 part 1 in the second case, and proposition 5.8 parts 2 and 3 in the third case (according as $\overline{\rho}$ is split or not); again we just do the third case, which is the most complicated. Equation (3) is equivalent to the equations:

$$\begin{aligned}\mathcal{C}(\overline{\rho}, \tau_{\zeta,s}) &= \mathcal{C}(\overline{\rho}, \mathbb{1}) &&= [\mathfrak{a}_{nr}]\\ \mathcal{C}(\overline{\rho}, \tau_{\zeta,ns}) &= \mathcal{C}(\overline{\rho}, \mathrm{St}) &&= [\mathfrak{a}_{nr}] + [\mathfrak{a}_N]\\ \mathcal{C}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}) &= \mathcal{C}(\overline{\rho}, \mathbb{1}) + \mathcal{C}(\overline{\rho}, \mathrm{St}) &&= [\mathfrak{a}_{nr}] + [\mathfrak{a}_{nr}] + [\mathfrak{a}_N]\end{aligned}$$

and

$$\mathcal{C}(\overline{\rho}, \tau) = 0$$

if $\tau \notin \bigcup_{\zeta,\zeta_1,\zeta_2}\{\tau_{\zeta,s}, \tau_{\zeta,ns}, \tau_{\zeta_1,\zeta_2}\}$. But by proposition 5.8 parts 2 and 3 we have:

$$\begin{aligned}\mathcal{C}(\overline{\rho}, \tau_{\zeta,s}) &= Z^4(\overline{R}(\overline{\rho}, \tau_{\zeta,s})) &&= [\mathfrak{a}_{nr}]\\ \mathcal{C}(\overline{\rho}, \tau_{\zeta,ns}) &= Z^4(\overline{R}(\overline{\rho}, \tau_{\zeta,s})) + Z^4(\overline{R}(\overline{\rho}, \tau_{1,ns})) &&= [\mathfrak{a}_{nr}] + [\mathfrak{a}_N]\\ \mathcal{C}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}) &= Z^4(\overline{R}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})) &&= 2[\mathfrak{a}_{nr}] + [\mathfrak{a}_N]\end{aligned}$$

and

$$\mathcal{C}(\overline{\rho}, \tau) = 0$$

if $\tau \notin \bigcup_{\zeta,\zeta_1,\zeta_2}\{\tau_{\zeta,s}, \tau_{\zeta,ns}, \tau_{\zeta_1,\zeta_2}\}$, as required. $\square$

Remark 4.3. Although the definition of $\mathcal{C}(\overline{\rho}, \tau)$ may seem ad-hoc, it in fact has the following natural interpretation: it is the reduction modulo λ of the cycle in $\mathcal{Z}^4(R^{\square}(\overline{\rho}))$ obtained by taking the Zariski closure of the closed points $x \in \operatorname{Spec} R^{\square}(\overline{\rho})[1/l]$ such that $\mathrm{rec}^{-1}(\rho_x)|_K$ contains $\sigma(\tau)$.

Remark 4.4. We conjecture that the theorem remains true when $l = 2$.

5. Calculations

Let $\overline{\rho} : G_F \to GL_2(\mathbb{F})$ be a continuous representation. The aims of this section are to give explicit presentations for the rings $R^{\square}(\overline{\rho}, \tau)$ and to compute the cycles $Z(\overline{R}^{\square}(\overline{\rho}, \tau)) \in \mathcal{Z}^4(\operatorname{Spec} \overline{R}^{\square}(\overline{\rho}))$. We continue to assume that E is sufficiently large, as defined at the start of the previous section.

5.1. **Simple cases.** When $\overline{\rho}|_{\tilde{P}_F}$ is not scalar, then lemma 2.3 allows us to determine the universal framed deformation rings. Recall that if $\overline{r} : I_F \to GL_2(\mathbb{F})$ is a representation that extends to G_F then we have defined the set $L(\overline{r})$ of types that lift $\overline{r}$.

Proposition 5.1. *If $\overline{\rho}|_{\tilde{P}_F}$ is irreducible, then*

$$R^{\square}(\overline{\rho}) \cong \mathcal{O}[[X, Y, Z_1, Z_2, Z_3]]/((1 + X)^{l^a} - 1).$$

The l^a irreducible components of $\operatorname{Spec} R^{\square}(\overline{\rho})$ *are precisely the* $\operatorname{Spec} R^{\square}(\overline{\rho}, \tau)$ *for $\tau \in L(\overline{\rho}|_{I_F})$.*

If $\overline{\rho}|_{\tilde{P}_F}$ is a sum of distinct characters which extend to G_F, then

$$R^{\square}(\overline{\rho}) \cong \mathcal{O}[[X_1, X_2, Y_1, Y_2, Z_1, Z_2]]/((1+X_1)^{l^a} - 1, (1+X_2)^{l^a} - 1).$$

The l^{2a} irreducible components of $\operatorname{Spec} R^{\square}(\overline{\rho})$ *are precisely the* $\operatorname{Spec} R^{\square}(\overline{\rho}, \tau)$ *for $\tau \in L(\overline{\rho}|_{I_F})$.*

If $\overline{\rho}|_{\tilde{P}_F}$ is a sum of distinct characters which are conjugate by the non-trivial element of $G_L \setminus G_F$, then

$$R^{\square}(\overline{\rho}) \cong \mathcal{O}[[X, Y, Z_1, Z_2, Z_3]]/((1+X)^{l^b} - 1).$$

The l^b irreducible components of $\operatorname{Spec} R^{\square}(\overline{\rho})$ *are precisely the* $\operatorname{Spec} R^{\square}(\overline{\rho}, \tau)$ *for $\tau \in L(\overline{\rho}|_{I_F})$.*

Proof. This follows straightforwardly from lemma 2.3. Suppose first that $\overline{\rho}|_{\tilde{P}_F}$ is irreducible. Then there is a unique irreducible representation θ of $\tilde{P}_F$ such that $\overline{\rho}_\theta$ (in the notation of lemma 2.3) is non-zero. For that θ, $\overline{\rho}_\theta$ is an unramified one-dimensional representation of G_F. So by lemmas 2.3 and 2.5:

$$R^{\square}(\overline{\rho}) \cong R^{\square}(\overline{\rho}_\theta)[[Z_1, Z_2, Z_3]] \cong \mathcal{O}[[X, Y, Z_1, Z_2, Z_3]]/((1+X)^{l^a} - 1).$$

We have $\rho^{\square} \cong \tilde{\theta} \otimes \chi^{\square}$ where $\chi^{\square}$ is the universal character $G_F \to R^{\square}(\overline{\rho}_\theta)^{\times}$.

Suppose now that $\overline{\rho}|_{\tilde{P}_F} = \theta_1 \oplus \theta_2$ for distinct characters θ_1 and θ_2. Suppose first that the θ_i are not G_F-conjugate. As in lemma 2.3, we pick $\mathcal{O}$-characters $\tilde{\theta}_1$ and $\tilde{\theta}_2$ of G_F lifting and extending θ_1 and θ_2. Then (in the notation of lemma 2.3) $\overline{\rho}_{\theta_1}$ and $\overline{\rho}_{\theta_2}$ are both unramified characters. By lemmas 2.3 and 2.5:

$$\begin{aligned} R^{\square}(\overline{\rho}) &\cong \left(R^{\square}(\overline{\rho}_{\theta_1}) \hat{\otimes} R^{\square}(\overline{\rho}_{\theta_2})\right)[[Z_1, Z_2]] \\ &\cong \mathcal{O}[[X_1, X_2, Y_1, Y_1, Z_1, Z_2]]/((1+X_1)^{l^a} - 1, (1+X_2)^{l^a} - 1). \end{aligned}$$

We have

$$\rho^{\square} \cong \tilde{\theta}_1 \otimes \chi_1^{\square} \oplus \tilde{\theta}_2 \otimes \chi_2^{\square}$$

where each $\chi_i^{\square}$ is the universal character over $R^{\square}(\overline{\rho}_{\theta_i})$.

Suppose finally that θ_1 and θ_2 are G_F-conjugate. We take $\theta = \theta_1$; then $G_\theta = G_L$ where L is a quadratic extension of F. In fact, since $\tilde{P}_F \subset G_L$ and l is odd, we must have that G_L is the unramified quadratic extension of F. As in lemma 2.3, pick an $\mathcal{O}$-character $\tilde{\theta}$ of G_L lifting and extending θ. Then (in the notation of lemma 2.3) $\overline{\rho}_\theta$ is an unramified character of G_L. By lemmas 2.3 and 2.5:

$$\begin{aligned} R^{\square}(\overline{\rho}) &\cong R^{\square}(\overline{\rho}_\theta)[[Z_1, Z_2, Z_3]] \\ &\cong \mathcal{O}[[X, Y, Z_1, Z_2, Z_3]]/((1+X)^{l^b} - 1), \end{aligned}$$

since $v_l(q^2-1) = l^b$. We have

$$\rho^{\square} \cong \operatorname{Ind}_{G_L}^{G_F}\left(\tilde{\theta} \otimes \chi^{\square}\right)$$

where $\chi^{\square}$ is the universal character over $R^{\square}(\overline{\rho}_\theta)$.

We show that $f : \operatorname{Spec}(R^{\square}(\overline{\rho}, \tau)) \mapsto \tau$ is a bijection from the set of irreducible components of $\operatorname{Spec}(R^{\square}(\overline{\rho}))$ to $L(\overline{\rho}|_{I_F})$. It is easy to see that f is an injection (from our explicit expressions for $\rho^{\square}$). The type of the $\overline{E}$-points of $\operatorname{Spec}(R^{\square}(\overline{\rho}, \tau))$ is constant on irreducible components, so to show that a particular τ is in the image of f it suffices to produce a lift of $\overline{\rho}$ to $\overline{E}$ of type τ. Each $\tau \in L(\overline{\rho}|_{I_F})$ is, by definition, the type of a lift of some $\overline{\rho}'$ with $\overline{\rho}'|_{I_F} \cong \overline{\rho}|_{I_F}$. But it is clear from the calculations above that the image of f only depends on $\overline{\rho}|_{I_F}$, and so f is surjective as required. □

Corollary 5.2. *If $\overline{\rho}|_{\tilde{P}_F}$ is not scalar, then $\overline{R}^{\square}(\overline{\rho})$ has a unique minimal prime $\mathfrak{a}$, which has dimension 4. For τ an inertial type we have that*

$$Z^4(\overline{R}^{\square}(\overline{\rho},\tau)) = [\mathfrak{a}]$$

if $\tau \in L(\overline{\rho}|_{\tilde{P}_F})$ and $Z^4(\overline{R}^{\square}(\overline{\rho},\tau)) = 0$ otherwise.

We may now assume that $\overline{\rho}|_{\tilde{P}_F}$ is scalar; after a twist (invoking [CHT08] lemma 2.4.11 to extend the character occurring in $\overline{\rho}|_{\tilde{P}_F}$ to the whole Galois group), we may assume that $\overline{\rho}|_{\tilde{P}_F}$ is *trivial*, so that any lift of $\overline{\rho}|_{\tilde{P}_F}$ is also trivial. In this case, then, $\overline{\rho}|_{I_F}$ is inflated from a representation of the (procyclic) pro-l group $I_F/\tilde{P}_F$ over a field of characteristic l. Any irreducible representation in characteristic l of an l-group is trivial, and so $\overline{\rho}|_{I_F}$ must be an extension of the trivial representation by the trivial representation. Now, because $\phi\sigma\phi^{-1} = \sigma^q$, $\overline{\rho}(\phi)$ maps the subspace of fixed vectors of $\overline{\rho}(\sigma)$ to itself; therefore $\overline{\rho}$ must be an extension of unramified characters. That is, there is a short exact sequence

$$0 \to \chi_1 \to \overline{\rho} \to \chi_2 \to 0$$

for unramified characters χ_1 and χ_2. Such an extension corresponds to an element of $H^1(G_F, \chi_1\chi_2^{-1})$; by a simple calculation with the local Euler characteristic formula and local Tate duality, this cohomology group is non-zero if and only if $\chi_1 = \chi_2$ or $\chi_1 = \chi_2\epsilon$. So we can easily deal with the case where neither of these two possibilities can occur.

Proposition 5.3. *Suppose that $\overline{\rho}|_{\tilde{P}_F}$ is trivial and that $\overline{\rho}(\phi)$ has eigenvalues $\overline{\alpha}, \overline{\beta} \in \mathbb{F}$ with $\overline{\alpha}/\overline{\beta} \notin \{1, q, q^{-1}\}$. Then*

$$R^{\square}(\overline{\rho}) \cong \frac{\mathcal{O}[[A,B,P,Q,X,Y]]}{((1+P)^{l^a}-1,(1+Q)^{l^a}-1)},$$

and $\rho^{\square}(\sigma)$ is diagonalizable with eigenvalues $1+P$ and $1+Q$.

For ζ an l^ath root of unity (possibly equal to 1), we have that

$$\begin{aligned} R^{\square}(\overline{\rho},\tau_{\zeta,s}) &= \mathcal{O}[[A,B,P,Q,X,Y]]/(1+P-\zeta, 1+Q-\zeta) \\ &\cong \mathcal{O}[[A,B,X,Y]] \end{aligned}$$

is formally smooth of relative dimension 4 over $\mathcal{O}$, and $R^{\square}(\overline{\rho},\tau_{\zeta,ns}) = 0$. If $q = 1$ mod *l and ζ_1, ζ_2 are distinct l^ath roots of unity, then*

$$\begin{aligned} R^{\square}(\overline{\rho},\tau_{\zeta_1,\zeta_2}) &= \frac{\mathcal{O}[[A,B,P,Q,X,Y]]}{(2+P+Q-\zeta_1-\zeta_2, PQ-(\zeta_1-1)(\zeta_2-1))} \\ &\cong \mathcal{O}[[A,B,P,X,Y]]/(1+P-\zeta_1)(1+P-\zeta_2). \end{aligned}$$

For all other τ, $R^{\square}(\overline{\rho},\tau) = 0$.

The ideal $\mathfrak{a}_{nr}$ defining $\overline{R}^{\square}(\overline{\rho},\tau_{1,s})$ is the unique minimal prime of $\overline{R}^{\square}(\overline{\rho})$. We have:

$$Z^4(\overline{R}^{\square}(\overline{\rho},\tau)) = \begin{cases} [\mathfrak{a}_{nr}] & \text{if } \tau = \tau_{\zeta,s} \\ 2[\mathfrak{a}_{nr}] & \text{if } \tau = \tau_{\zeta_1,\zeta_2} \\ 0 & \text{if } \tau = \tau_{\zeta,ns}. \end{cases}$$

Proof. First note that, by the above cohomology calculation, $\overline{\rho}(\sigma)$ must be trivial.

Let α and β be lifts of $\overline{\alpha}$ and $\overline{\beta}$ to $\mathcal{O}$. Suppose that $\mathcal{A}$ is an object of $\mathcal{C}_{\mathcal{O}}$ and that M is a free $\mathcal{A}$-module of rank 2 with a continuous action of G_F given by $\rho : G_F \to \mathrm{Aut}_{\mathcal{A}}(M)$, reducing to $\overline{\rho}$ modulo $\mathfrak{m}_{\mathcal{A}}$. Suppose that the characteristic polynomial of $\rho(\phi)$ is $(X-\alpha-A)(X-\beta-B)$, where $A, B \in \mathfrak{m}_{\mathcal{A}}$ – note that by Hensel's lemma the characteristic polynomial *does* have roots in $\mathcal{A}$ reducing to $\overline{\alpha}$ and $\overline{\beta}$. Then there is a decomposition

$$M = (\rho(\phi) - \alpha - A)M \oplus (\rho(\phi) - \beta - B)M.$$

Here it is crucial that $\alpha+A$, $\beta+B$ and $\alpha-\beta+A-B$ are all invertible in $\mathcal{A}$. If $\overline{v}_\alpha, \overline{v}_\beta$ is a basis of eigenvectors of $\overline{\rho}(\phi)$ in $M \otimes \mathbb{F}$ and v_α, v_β is a basis of M lifting $\overline{v}_\alpha, \overline{v}_\beta$ then there are unique $X, Y \in \mathfrak{m}_{\mathcal{A}}$ such that $v_\alpha + Xv_\beta, v_\beta + Yv_\alpha$ are eigenvectors of $\rho(\phi)$. Moreover, replacing (v_α, v_β) by $(\mu v_\alpha, \mu v_\beta)$ for $\mu \in 1 + \mathfrak{m}_{\mathcal{A}}$ does not change X and Y.

Therefore we may assume that $\overline{\rho}(\phi) = \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \overline{\beta} \end{pmatrix}$ and that

$$\begin{aligned} \rho(\phi) &= \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha + A & 0 \\ 0 & \beta + B \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix} \\ \rho(\sigma) &= \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1+P & R \\ S & 1+Q \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix} \end{aligned}$$

where $X, Y, P, R, S, Q \in \mathfrak{m}_{\mathcal{A}}$ are uniquely determined by ρ. The equation $\phi\sigma\phi^{-1} = \sigma^q$ implies that

$$\begin{pmatrix} \alpha + A & 0 \\ 0 & \beta + B \end{pmatrix} \begin{pmatrix} 1+P & R \\ S & 1+Q \end{pmatrix} \begin{pmatrix} \alpha + A & 0 \\ 0 & \beta + B \end{pmatrix}^{-1} = \begin{pmatrix} 1+P & R \\ S & 1+Q \end{pmatrix}^{q}.$$

Looking at the top right and bottom left entries gives that $R = S = 0$. Then looking at the diagonal entries gives that $(1+P)^{q-1} = (1+Q)^{q-1} = 1$, which is equivalent to $(1+P)^{l^a} = (1+Q)^{l^a} = 1$. Thus

$$R^{\square}(\overline{\rho}) \cong \frac{\mathcal{O}[[A,B,P,Q,X,Y]]}{((1+P)^{l^a}-1, (1+Q)^{l^a}-1)}.$$

The possible inertial types are $\tau_{\zeta,s}$ and τ_{ζ_1,ζ_2} ($\tau_{\zeta,ns}$ cannot occur since all lifts are diagonalisable). Clearly $R^{\square}(\overline{\rho}, \tau_{\zeta,s})$ is defined by the equations $1+P = 1+Q = \zeta$. The ring $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})^{\circ}$ is cut out by the equations $2+P+Q = \zeta_1+\zeta_2$, $(1+P)(1+Q) = \zeta_1\zeta_2$ and the redundant equations $(1+P)^{l^a} = (1+Q)^{l^a} = 1$. But

$$R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})^{\circ} \cong \mathcal{O}[[A,B,P,X,Y]]/((1+P-\zeta_1)(1+P-\zeta_2))$$

is reduced and λ-torsion free and so is equal to $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})$.

For the reduction modulo λ, simply note that:

$$\begin{aligned} \overline{R}^{\square}(\overline{\rho}) &= \mathbb{F}[[A,B,P,Q,X,Y]]/(P^{l^a}, Q^{l^a}) \\ \overline{R}^{\square}(\overline{\rho}, \tau_{\zeta,s}) &= \mathbb{F}[[A,B,P,Q,X,Y]]/(P,Q) \end{aligned}$$

and

$$\overline{R}^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}) = \mathbb{F}[[A,B,P,Q,X,Y]]/(P^2, Q^2, P+Q).$$

So $\mathfrak{a}_{nr} = (P,Q)$ is the unique minimal prime of $\overline{R}^{\square}(\overline{\rho})$ and the multiplicities are as claimed. $\square$

We extract one part of the proof of this proposition for future use:

Lemma 5.4. *If $\overline{\rho}(\phi)$ has distinct eigenvalues, we may assume that it is diagonal. In that case, there exists a unique matrix $\begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix} \in GL_2(R^{\square}(\overline{\rho}))$, reducing to the identity modulo the maximal ideal, such that $\rho^{\square}(\phi) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \Phi \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}$ for a* diagonal *matrix Φ.*

Proof. This is simply the first half of the proof of the previous proposition. $\square$

5.2. $q \neq \pm 1 \bmod l$**.** Suppose that $q \neq \pm 1 \bmod l$. By lemma 5.3, we have already dealt with the cases in which the eigenvalues of $\overline{\rho}(\phi)$ are not in the ratio 1 or $q^{\pm 1}$. All other cases are dealt with by the following (after twisting and conjugating $\overline{\rho}$). Note that, by lemma 3.7, the only possible types when $\overline{\rho}|_{\tilde{P}_F}$ is trivial are $\tau_{1,s}$ and $\tau_{1,ns}$.

Proposition 5.5. *Suppose that $q \neq \pm 1 \mod l$, and that $\overline{\rho}|_{\tilde{P}_F}$ is trivial. Then*

1. *Suppose that $\overline{\rho}(\sigma)$ is trivial, and that $\overline{\rho}(\phi) = \begin{pmatrix} 1 & \overline{y} \\ 0 & 1 \end{pmatrix}$ for $\overline{y} \in \mathbb{F}$. Then $R^{\square}(\overline{\rho}, \tau_{1,s}) = R^{\square}(\overline{\rho})$ is formally smooth of relative dimension 4 over $\mathcal{O}$, while $R^{\square}(\overline{\rho}, \tau_{1,ns}) = 0$.*
2. *Suppose that $\overline{\rho}(\sigma) = \begin{pmatrix} 1 & \overline{x} \\ 0 & 1 \end{pmatrix}$ and $\overline{\rho}(\phi) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$.*

   If $\overline{x} \neq 0$, then $R^{\square}(\overline{\rho}, \tau_{1,ns}) = R^{\square}(\overline{\rho})$ is formally smooth of relative dimension 4 over $\mathcal{O}$, while $R^{\square}(\overline{\rho}, \tau_{1,s}) = 0$.

   If $\overline{x} = 0$ then

   $$R^{\square}(\overline{\rho}) \cong \mathcal{O}[[X_1, \dots, X_5]]/(X_1 X_2).$$

   The quotients by the two minimal primes are $R^{\square}(\overline{\rho}, \tau_{1,s})$ and $R^{\square}(\overline{\rho}, \tau_{1,ns})$, so that both are formally smooth of relative dimension 4 over $\mathcal{O}$. The minimal primes $\mathfrak{a}_{nr}$ and $\mathfrak{a}_N$ of $\overline{R}^{\square}(\overline{\rho})$ which respectively define $\overline{R}^{\square}(\overline{\rho}, \tau_{1,s})$ and $\overline{R}^{\square}(\overline{\rho}, \tau_{1,ns})$ are distinct.

Proof. For the first part, write

$$\rho^{\square}(\sigma) = \begin{pmatrix} 1+A & B \\ C & 1+D \end{pmatrix}$$
$$\rho^{\square}(\phi) = \begin{pmatrix} 1+P & y+R \\ S & 1+Q \end{pmatrix}$$

where y is a lift of $\overline{y}$ (taken to be zero if $\overline{y} = 0$) and $A, B, C, D, P, Q, R, S \in \mathfrak{m}$.

Let $I = (A, B, C, D)$. Considering the equation $\rho^{\square}(\phi)\rho^{\square}(\sigma) = \rho^{\square}(\sigma)^q \rho^{\square}(\phi)$ modulo the ideal $I\mathfrak{m}$ gives equations $Cy \equiv (q-1)A$, $B + Dy \equiv qAy + qB$, $C \equiv qC$ and $(q-1)D + qCy \equiv 0$, all modulo $I\mathfrak{m}$. As $q \neq 1 \mod l$ we find that $I = I\mathfrak{m}$. Therefore, by Nakayama's lemma, $I = 0$ and $\rho^{\square}$ is unramified. So $R^{\square}(\overline{\rho}) = R^{\square}(\overline{\rho}, \tau_{1,s}) \cong \mathcal{O}[[P, Q, R, S]]$ as claimed. Note that this proof is still valid if $q = -1 \mod l$.

The proof of the second part is similar. By lemma 5.4, we may write

$$\rho^\square(\sigma) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1+A & x+B \\ C & 1+D \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}$$
$$\rho^\square(\phi) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}^{-1} \begin{pmatrix} q(1+P) & 0 \\ 0 & 1+Q \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}$$

with x a lift of $\overline{x}$ (taken to be zero if $\overline{x} = 0$) and $A, B, C, D, X, Y, P, Q \in \mathfrak{m}$.

Let $I = (A, C, D)$. Considering the relation $\phi\sigma\phi^{-1} = \sigma^q$ modulo $I\mathfrak{m}$ and applying Nakayama's lemma as before now yields $A = C = D = 0$ (using that $q^2 \neq 1 \bmod l$). The relation (not modulo any ideal) gives that $(x+B)(P-Q) = 0$, and it is easy to see if this equality holds then the given formulae for $\rho^\square$ do indeed define a representation so that

$$R^\square(\overline{\rho}) = \frac{\mathcal{O}[[B, P, Q, X, Y]]}{((x+B)(P-Q))}.$$

If $\overline{x} \neq 0$ then this implies that $P = Q$. Then $R^\square(\overline{\rho}) = \mathcal{O}[[B, P, X, Y]]$. It is clear that $R^\square(\overline{\rho}) = R^\square(\overline{\rho}, \tau_{1,ns})$, and the proposition follows.

If $\overline{x} = 0$ then, writing $U = P - Q$, we have $R^\square(\overline{\rho}) = \mathcal{O}[[B, P, U, X, Y]]/(BU)$. In these coordinates, it is clear from the description of $\rho^\square$ that

$$R^\square(\overline{\rho}, \tau_{1,s}) = R^\square(\overline{\rho})/(B)$$

and

$$R^\square(\overline{\rho}, \tau_{1,ns}) = R^\square(\overline{\rho})/(U).$$

The proposition follows. □

5.3. $q = -1 \bmod l$. Suppose that $q = -1 \bmod l$. By proposition 5.3, we have already dealt with the cases in which the eigenvalues of $\overline{\rho}(\phi)$ are not in the ratio 1 or -1. All other cases are dealt with by the following (after twisting and conjugating $\overline{\rho}$). By lemma 3.7, the only possible types when $\overline{\rho}|_{\tilde{P}_F}$ is trivial are $\tau_{1,s}$, $\tau_{1,ns}$ and τ_ξ for ξ a non-trivial l^bth root of unity.

Proposition 5.6. *Suppose that $q = -1 \bmod l$ and that $\overline{\rho}|_{\tilde{P}_F}$ is trivial.*

(1) *Suppose that $\overline{\rho}(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\overline{\rho}(\phi) = \begin{pmatrix} 1 & \overline{y} \\ 0 & 1 \end{pmatrix}$ for $\overline{y} \in \mathbb{F}$. Then*

$$R^\square(\overline{\rho}, \tau_{1,s}) = R^\square(\overline{\rho})$$

is formally smooth of relative dimension 4 over $\mathcal{O}$, while

$$R^\square(\overline{\rho}, \tau_{1,ns}) = R^\square(\overline{\rho}, \tau_\xi) = 0.$$

If $\mathfrak{a}_{nr}$ is the unique minimal prime of $\overline{R}^\square(\overline{\rho})$, then we have

$$Z^4(\overline{R}^\square(\overline{\rho}, \tau)) = \begin{cases} [\mathfrak{a}_{nr}] & \text{if } \tau = \tau_{1,s} \\ 0 & \text{if } \tau = \tau_{1,ns} \\ 0 & \text{if } \tau = \tau_\xi. \end{cases}$$

(2) *Suppose that $\overline{\rho}(\sigma) = \begin{pmatrix} 1 & \overline{x} \\ 0 & 1 \end{pmatrix}$ and $\overline{\rho}(\phi) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ for $\overline{x} \in \mathbb{F}$.*

(a) *If $\overline{x} \neq 0$, then $R^{\square}(\overline{\rho}, \tau_{1,ns})$ and $R^{\square}(\overline{\rho}, \tau_{\xi})$ are formally smooth of relative dimension 4 over $\mathcal{O}$, while $R^{\square}(\overline{\rho}, \tau_{1,s}) = 0$. If $\mathfrak{a}_N$ is the prime ideal of $\overline{R}^{\square}(\overline{\rho})$ cutting out $\overline{R}^{\square}(\overline{\rho}, \tau_{1,ns})$ then we have*

$$Z^4(\overline{R}^{\square}(\overline{\rho}, \tau)) = \begin{cases} 0 & \text{if } \tau = \tau_{1,s} \\ [\mathfrak{a}_N] & \text{if } \tau = \tau_{1,ns} \\ [\mathfrak{a}_N] & \text{if } \tau = \tau_{\xi}. \end{cases} \tag{4}$$

(b) *If $\overline{x} = 0$, then $R^{\square}(\overline{\rho}, \tau_{1,s})$ is formally smooth of relative dimension 4 over $\mathcal{O}$ and*

$$R^{\square}(\overline{\rho}, \tau_{1,ns}) \cong \frac{\mathcal{O}[[X_1, \ldots, X_6]]}{((X_1, X_3) \cap (X_2, X_3 - (q+1)))}$$

is a non-Cohen–Macaulay ring of relative dimension 4 over $\mathcal{O}$. Its spectrum is the scheme theoretic union of two formally smooth components that do not intersect in the generic fibre. Lastly,

$$R^{\square}(\overline{\rho}, \tau_{\xi}) \cong \frac{\mathcal{O}[[X_1, \ldots, X_5]]}{(X_1 X_2 - (\xi - \xi^{-1})^2)}$$

is a complete intersection domain of relative dimension 4 over $\mathcal{O}$ with formally smooth generic fibre. If $\mathfrak{a}_{nr}$ is the prime of $\overline{R}^{\square}(\overline{\rho})$ corresponding to $\overline{R}^{\square}(\overline{\rho}, \tau_{1,s})$ and $\mathfrak{a}_N, \mathfrak{a}'_N$ are the prime ideals of $\overline{R}^{\square}(\overline{\rho})$ corresponding to the two minimal primes of $\overline{R}^{\square}(\overline{\rho}, \tau_{1,ns})$, then we have

$$Z^4(\overline{R}^{\square}(\overline{\rho}, \tau)) = \begin{cases} [\mathfrak{a}_{nr}] & \text{if } \tau = \tau_{1,s} \\ [\mathfrak{a}_N] + [\mathfrak{a}_{N'}] & \text{if } \tau = \tau_{1,ns} \\ [\mathfrak{a}_N] + [\mathfrak{a}_{N'}] & \text{if } \tau = \tau_{\xi}. \end{cases} \tag{5}$$

Proof. The proof of the first part is identical to that of proposition 5.5, part 1.

For the second part, by lemma 5.4 we may write

$$\rho^{\square}(\sigma) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix} \begin{pmatrix} 1+A & x+B \\ C & 1+D \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}$$
$$\rho^{\square}(\phi) = \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix} \begin{pmatrix} -(1+P) & 0 \\ 0 & 1+Q \end{pmatrix} \begin{pmatrix} 1 & X \\ Y & 1 \end{pmatrix}$$

with x a lift of $\overline{x}$ (taken to be zero if $\overline{x} = 0$) and $A, B, C, D, X, Y, P, Q \in \mathfrak{m}$.

Firstly, it is clear that $R^{\square}(\overline{\rho}, \tau_{1,s}) = 0$ if $\overline{x} \neq 0$ and

$$R^{\square}(\overline{\rho}, \tau_{1,s}) \cong \mathcal{O}[[P, Q, X, Y]]$$

if $\overline{x} = 0$.

Next we deal with $\tau_{1,ns}$. On $R^{\square}(\overline{\rho}, \tau_{1,ns})$ we have the equations

$$\begin{aligned} \operatorname{tr}(\rho^{\square}(\sigma)) &= 2 \\ \det(\rho^{\square}(\sigma)) &= 1 \\ q \operatorname{tr}(\rho(\phi))^2 &= (q+1)^2 \det(\rho(\phi)) \end{aligned}$$

and

$$\rho^{\square}(\phi)\rho^{\square}(\sigma)\rho^{\square}(\phi)^{-1} = \rho^{\square}(\sigma)^q.$$

The first two of these may be rewritten as

$$A = -D$$

and

$$A^2 + (x+B)(C) = 0$$

and the third can be written as

$$(q+1+P+qQ)(q+1+Q+qP) = 0.$$

By the Cayley–Hamilton theorem, $(\rho^\square(\sigma)-1)^2 = 0$ on $R^\square(\overline{\rho}, \tau_{1,ns})^\circ$; it follows that $\rho^\square(\sigma)^q - 1 = q(\rho^\square(\sigma)-1)$ on $R^\square(\overline{\rho}, \tau_{1,ns})^\circ$ and so the relation $\phi\sigma\phi^{-1} = \sigma^q$ together with $D = -A$ yields the equation:

$$\begin{pmatrix} A & -(x+B)\frac{1+P}{1+Q} \\ -C\frac{1+Q}{1+P} & -A \end{pmatrix} = \begin{pmatrix} qA & q(x+B) \\ qC & -qA \end{pmatrix}.$$

Equating coefficients and using that 2 and $q-1$ are invertible we obtain that $A = D = 0$ and that

$$(x+B)(q+1+qQ+P) = 0 \tag{6}$$

$$C(q+1+Q+qP) = 0 \tag{7}$$

$$(x+B)C = 0 \tag{8}$$

$$(q+1+Q+qP)(q+1+qQ+P) = 0 \tag{9}$$

is a complete set of equations cutting out $R^\square(\overline{\rho}, \tau_{1,ns})^\circ$ (the last two equations being, respectively, the conditions on $\det(\rho^\square(\sigma))$ and on $\rho^\square(\phi)$).

If $\overline{x} \neq 0$ then these equations are equivalent to $q+1+qQ+P = 0$ and $C = 0$ and so we see that

$$R^\square(\overline{\rho}, \tau_{1,ns}) \cong \mathcal{O}[[B, P, X, Y]].$$

If $\overline{x} = 0$ then the left hand sides of the four equations given generate the ideal

$$I = (B, q+1+Q+qP) \cap (C, q+1+qQ+P)$$

in $\mathcal{O}[[B, C, P, Q, X, Y]]$. Since $\mathcal{O}[[B, C, P, Q, X, Y]]/I$ is reduced and λ-torsion free and a Zariski dense set of its $\overline{E}$-points have type $\tau_{1,ns}$, it is equal to $R^\square(\overline{\rho}, \tau_{1,ns})$. After the change of variables $X_3 = \frac{q(q+1+Q+qP)}{(q-1)(1+P)}$, $(X_1, X_2, X_4, X_5, X_6) = (B, C, P, X, Y)$ we get the presentation given in the proposition.

Let

$$\mathcal{S} = \frac{\mathcal{O}[[X_1, X_2, X_3]]}{(X_1, X_3) \cap (X_2, X_3 - (q+1))}.$$

Then $\mathcal{S}$ has dimension two. We show that $\mathcal{S}$ is not Cohen–Macaulay; the same is then true for $R^\square(\overline{\rho}, \tau_{1,ns})$. Now, λ is a non-zerodivisor in $\mathcal{S}$, and

$$\mathcal{S}/\lambda = \frac{\mathbb{F}[[X_1, X_2, X_3]]}{(X_1X_2, X_1X_3, X_2X_3, X_3^2)}.$$

The maximal ideal of $\mathcal{S}/\lambda$ is annihilated by X_3, and $X_3 \neq 0$ in $\mathcal{S}/\lambda$. So $\mathcal{S}/\lambda$, and hence $\mathcal{S}$, is not Cohen–Macaulay. The remaining statements about $R^\square(\overline{\rho}, \tau_{1,ns})$ are clear.

Now suppose that $\tau = \tau_\xi$. On $R^\square(\overline{\rho}, \tau_\xi)$ we have

$$\mathrm{tr}(\rho^\square(\sigma)) = \xi + \xi^{-1}$$
$$\det(\rho^\square(\sigma)) = 1$$

and

$$\rho^\square(\phi)\rho^\square(\sigma)\rho^\square(\phi)^{-1} = \rho^\square(\sigma)^q.$$

The first two of these may be rewritten as

$$A + D = \xi + \xi^{-1} - 2$$

and

$$AD - (x + B)C = 2 - \xi - \xi^{-1}.$$

By the Cayley–Hamilton theorem, $(\rho^{\square}(\sigma) - \xi)(\rho^{\square}(\sigma) - \xi^{-1}) = 0$. As

$$T^q \equiv \xi + \xi^{-1} - T \mod (T - \xi)(T - \xi^{-1})$$

in $\mathbb{Z}[T]$, the relation $\phi\sigma\phi^{-1} = \sigma^q$ yields

$$\begin{pmatrix} 1 + A & -(x + B)\frac{1+P}{1+Q} \\ -C\frac{1+Q}{1+P} & 1 + D \end{pmatrix} = \begin{pmatrix} \xi + \xi^{-1} - 1 - A & -(x + B) \\ -C & \xi + \xi^{-1} - 1 - D \end{pmatrix}.$$

Equating coefficients and combining with the equation $\det(\rho^{\square}(\sigma)) = 1$ we get:

$$A = D = \frac{\xi + \xi^{-1}}{2} - 1 \tag{10}$$

$$(x + B)(P - Q) = 0 \tag{11}$$

$$C(P - Q) = 0 \tag{12}$$

$$4(x + B)C = (\xi - \xi^{-1})^2. \tag{13}$$

If $\overline{x} \neq 0$ then these equations are equivalent to $P = Q$ and $C = \frac{(\xi - \xi^{-1})^2}{4(x+B)}$, so that

$$R^{\square}(\overline{\rho}, \tau_\xi) \cong \mathcal{O}[[X, Y, B, P]].$$

If $\overline{x} = 0$, then the equations imply that

$$0 = BC(P - Q) = \left(\frac{\xi - \xi^{-1}}{2}\right)^2 (P - Q)$$

and hence that $P = Q$, as $R^{\square}(\overline{\rho}, \tau_\xi)$ is λ-torsion free by definition. Thus

$$R^{\square}(\overline{\rho}, \tau_\xi) \cong \frac{\mathcal{O}[[X, Y, B, C, P]]}{(4BC - (\xi - \xi^{-1})^2)}.$$

The remaining statements about $R^{\square}(\overline{\rho}, \tau_\xi)$ are clear.

Now we calculate the various $Z^4(\overline{R}^{\square}(\overline{\rho}, \tau))$. For part 1, this is trivial. For part 2, we have computed each $\overline{R}^{\square}(\overline{\rho}, \tau)$ as a quotient of the ring $\mathbb{F}[[A, B, C, D, P, Q, X, Y]]$ by an ideal which we call $I(\tau)$. We see that if $\overline{x} \neq 0$ then $I(\tau_{1,ns}) = I(\tau_\xi)$, and $\overline{R}^{\square}(\overline{\rho}, \tau_{1,s}) = 0$, from which equation 4 follows. If $\overline{x} = 0$ then

$$I(\tau_{1,s}) = (A, B, C, D)$$
$$I(\tau_{1,ns}) = (A, D, BC, B(Q - P), C(Q - P), (Q - P)^2)$$

and

$$I(\tau_\xi) = (A, D, BC, Q - P).$$

The minimal primes above these $I(\tau)$ in $\mathbb{F}[[A, \ldots, Y]]$ are $\mathfrak{a}_{nr} = (A, B, C, D)$, $\mathfrak{a}_N = (A, C, D, Q - P)$ and $\mathfrak{a}_{N'} = (A, B, D, Q - P)$; the multiplicities in equation 5 are then easily verified. $\square$

Remark 5.7. When $\overline{\rho}$ is unramified and $\overline{\rho}(\phi) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, the ring $R^{\square}(\overline{\rho}, \tau_{1,ns})$ is not Cohen–Macaulay. However the ring $R^{\square}(\overline{\rho}, \text{unip})$, defined to be the maximal reduced quotient of $R^{\square}(\overline{\rho})$ on which $\rho^{\square}(\sigma)$ is unipotent (so that $\operatorname{Spec} R^{\square}(\overline{\rho}, \text{unip})$ is the scheme-theoretic union of $\operatorname{Spec} R^{\square}(\overline{\rho}, \tau_{1,s})$ and $\operatorname{Spec} R^{\square}(\overline{\rho}, \tau_{1,ns})$ in $\operatorname{Spec} R^{\square}(\overline{\rho})$), *is* Cohen–Macaulay. Indeed it is easy to see from the above proof that

$$R^{\square}(\overline{\rho}, \text{unip}) \cong \frac{\mathcal{O}[[X_1, \ldots, X_6]]}{(X_1X_2, X_1(X_3 - (q+1)), X_2X_3)}$$

which is Cohen–Macaulay ($(\lambda, X_1 + X_2 + X_3, X_4, X_5, X_6)$ is a regular sequence).

5.4. $q = 1 \bmod l$**.** Suppose that $q = 1 \bmod l$. By proposition 5.3, we have already dealt with the cases in which the eigenvalues of $\overline{\rho}(\phi)$ are distinct. All other cases are dealt with by the following (after twisting and conjugating $\overline{\rho}$). Note that by lemma 3.7, the only possible types when $\overline{\rho}|_{\tilde{P}_F}$ is trivial are $\tau_{\zeta,s}$, $\tau_{\zeta,ns}$ and τ_{ζ_1,ζ_2} for ζ any l^ath root of unity and ζ_1, ζ_2 any distinct l^ath roots of unity.

Proposition 5.8. *Suppose that $q = 1 \bmod l$ and that $\overline{\rho}|_{\tilde{P}_F}$ is trivial. Suppose that $\overline{\rho}(\sigma) = \begin{pmatrix} 1 & \overline{x} \\ 0 & 1 \end{pmatrix}$ and $\overline{\rho}(\phi) = \begin{pmatrix} 1 & \overline{y} \\ 0 & 1 \end{pmatrix}$ for $\overline{x}$, $\overline{y} \in \mathbb{F}$.*

(1) *If $\overline{x} \neq 0$ then $R^{\square}(\overline{\rho}, \tau_{\zeta,s}) = 0$, while $R^{\square}(\overline{\rho}, \tau_{\zeta,ns})$ and $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})$ are formally smooth over $\mathcal{O}$ of relative dimension 4.*

If $\mathfrak{a}_N$ is the four-dimensional prime of $\overline{R}^{\square}(\overline{\rho})$ corresponding to $\overline{R}^{\square}(\overline{\rho}, \tau_{1,ns})$ then we have:

$$Z^4(\overline{R}^{\square}(\overline{\rho}, \tau)) = \begin{cases} 0 & \text{if } \tau = \tau_{\zeta,s} \\ [\mathfrak{a}_N] & \text{if } \tau = \tau_{\zeta,ns} \\ [\mathfrak{a}_N] & \text{if } \tau = \tau_{\zeta_1,\zeta_2}. \end{cases} \tag{14}$$

(2) *If $\overline{x} = 0$ and $\overline{y} \neq 0$, then $R^{\square}(\overline{\rho}, \tau_{\zeta,s})$ and $R^{\square}(\overline{\rho}, \tau_{\zeta,ns})$ are formally smooth over $\mathcal{O}$ of relative dimension 4 while*

$$R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}) \cong \mathcal{O}[[X_1, \ldots, X_5]]/(X_1^2X_2 - (\zeta_1 - \zeta_2)^2)$$

is a complete intersection domain of relative dimension 4 over $\mathcal{O}$.

If $\mathfrak{a}_{nr}$ and $\mathfrak{a}_N$ are the prime ideals of $\overline{R}^{\square}(\overline{\rho})$ corresponding to $\overline{R}^{\square}(\overline{\rho}, \tau_{1,s})$ and $\overline{R}^{\square}(\overline{\rho}, \tau_{1,ns})$ respectively, then

$$Z^4(\overline{R}^{\square}(\overline{\rho}, \tau)) = \begin{cases} [\mathfrak{a}_{nr}] & \text{if } \tau = \tau_{\zeta,s} \\ [\mathfrak{a}_N] & \text{if } \tau = \tau_{\zeta,ns} \\ 2[\mathfrak{a}_{nr}] + [\mathfrak{a}_N] & \text{if } \tau = \tau_{\zeta_1,\zeta_2}. \end{cases} \tag{15}$$

(3) *If $\overline{x} = \overline{y} = 0$, then $R^{\square}(\overline{\rho}, \tau_{\zeta,s})$ is formally smooth over $\mathcal{O}$ of relative dimension 4, $R^{\square}(\overline{\rho}, \tau_{\zeta,ns})$ is a non-Gorenstein Cohen–Macaulay domain of relative dimension 4 over $\mathcal{O}$, while $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})$ is a non-Gorenstein Cohen–Macaulay domain of relative dimension 4 over $\mathcal{O}$.*

Both $\overline{R}^{\square}(\overline{\rho}, \tau_{\zeta,s})$ and $\overline{R}^{\square}(\overline{\rho}, \tau_{\zeta,ns})$ are domains; let the corresponding primes of $\overline{R}^{\square}(\overline{\rho})$ be $\mathfrak{a}_{nr}$ and $\mathfrak{a}_N$ respectively. Then

$$Z^4(\overline{R}^{\square}(\overline{\rho}, \tau)) = \begin{cases} [\mathfrak{a}_{nr}] & \text{if } \tau = \tau_{\zeta,s} \\ [\mathfrak{a}_N] & \text{if } \tau = \tau_{\zeta,ns} \\ 2[\mathfrak{a}_{nr}] + [\mathfrak{a}_N] & \text{if } \tau = \tau_{\zeta_1,\zeta_2}. \end{cases} \tag{16}$$

Proof. Write

$$\rho^{\square}(\sigma) = \begin{pmatrix} 1+A & x+B \\ C & 1+D \end{pmatrix}$$
$$\rho^{\square}(\phi) = \begin{pmatrix} 1+P & y+R \\ S & 1+Q \end{pmatrix}$$

with $A, B, C, D, P, Q, R, S \in \mathfrak{m}$ and x, y lifts of $\overline{x}, \overline{y}$ (taken to be zero if $\overline{x}$ or $\overline{y} = 0$).

First, we have that $R^{\square}(\overline{\rho}, \tau_{\zeta,s}) = 0$ if $\overline{x} \neq 0$ and

$$R^{\square}(\overline{\rho}, \tau_{\zeta,s}) \cong \mathcal{O}[[P, Q, R, S]]$$

otherwise.

Next, we look at $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})$ for ζ_1 and ζ_2 distinct l^ath roots of unity. The condition that $\rho^{\square}(\sigma)$ has characteristic polynomial $(t-\zeta_1)(t-\zeta_2)$ is equivalent to the equations

$$A + D = \zeta_1 + \zeta_2 - 2$$

and

$$AD - (x+B)C = (\zeta_1 - 1)(\zeta_2 - 1).$$

Since $(t-\zeta_1)(t-\zeta_2) \mid t^{q-1} - 1$, by the Cayley–Hamilton theorem we have

$$\rho^{\square}(\sigma)^q = \rho^{\square}(\sigma)$$

on $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})^{\circ}$. So the relation $\phi\sigma\phi^{-1} = \sigma^q$ yields:

$$\begin{pmatrix} 1+A & x+B \\ C & 1+D \end{pmatrix} \begin{pmatrix} 1+P & y+R \\ S & 1+Q \end{pmatrix} = \begin{pmatrix} 1+P & y+R \\ S & 1+Q \end{pmatrix} \begin{pmatrix} 1+A & x+B \\ C & 1+D \end{pmatrix}.$$

Equating coefficients, eliminating D and writing $U = P - Q$ and

$$F = A - D = 2A - (\zeta_1 + \zeta_2 - 2)$$

we see that $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})$ is the reduced, l-torsion–free quotient of $\mathcal{O}[[B, C, F, P, R, S, U]]$ by the relations:

$$(x+B)S = (y+R)C \tag{17}$$
$$F(y+R) = U(x+B) \tag{18}$$
$$FS = UC \tag{19}$$
$$(\zeta_1 - \zeta_2)^2 = F^2 + 4(x+B)C. \tag{20}$$

If $\overline{x} \neq 0$ then these equations are equivalent to $U = F(y+R)(x+B)^{-1}$, $C = \frac{1}{4}\left((\zeta_1-\zeta_2)^2 - F^2\right)(x+B)^{-1}$ and $S = C(y+R)(x+B)^{-1}$, so that

$$R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}) \cong \mathcal{O}[[B, F, P, R]].$$

If $\overline{x} = 0$ and $\overline{y} \neq 0$, then $F = BU(y+R)^{-1}$ and $C = BS(y+R)^{-1}$ will be a solution to the equations (17) to (20) provided that

$$(\zeta_1 - \zeta_2)^2 = \left(\frac{B}{y+R}\right)^2 (U^2 + 4(y+R)S);$$

writing $(X_1, \ldots, X_5) = (B(y+R)^{-1}, U^2 + 4(y+R)S, P, R, U)$ we get

$$R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}) \cong \frac{\mathcal{O}[[X_1, \ldots, X_5]]}{X_1^2 X_2 - (\zeta_1 - \zeta_2)^2}$$

as claimed. The other statements about $R^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2})$ follow easily.

If $\overline{x} = \overline{y} = 0$, then let $\mathcal{A} = \mathcal{O}[[B, C, F, P, R, S, U]]$ and $I \lhd \mathcal{A}$ be the ideal:

$$I = \left((\zeta_1 - \zeta_2)^2 - F^2 - 4BC, BS - CR, FR - BU, FS - CU\right).$$

Note that the ideal

$$J = (BS - CR, FR - BU, FS - CU)$$

is generated by the 2×2 minors of $\begin{pmatrix} B & C & F \\ R & S & U \end{pmatrix}$. So, by proposition 2.7, $\mathcal{A}/J$ is a Cohen–Macaulay, non-Gorenstein domain. Since $F^2 - 4BC$ is not zero in the domain $\mathcal{A}/J \otimes \mathbb{F}$, $(\lambda, F^2 - 4BC)$ is a regular sequence in $\mathcal{A}/J$. Hence $(F^2 - 4BC - (\zeta_1 - \zeta_2)^2, \lambda)$ is a regular sequence in $\mathcal{A}/J$, and therefore $\mathcal{A}/I$ is $\mathcal{O}$-flat, Cohen–Macaulay and non–Gorenstein. It is reduced because it is Cohen–Macaulay and, as we shall show in the next paragraph, generically reduced.

To show that $\mathcal{A}/I$ is irreducible, it suffices to show that $\mathcal{X} = \operatorname{Spec}(\mathcal{A}/I \otimes E)$ is irreducible. This follows if we can show that $\mathcal{X}$ is formally smooth and connected. As $F^2 - 4BC \neq 0$ on $\mathcal{X}$, it is covered by the affine open subsets $\mathcal{U}_B = \{B \neq 0\}$ and $\mathcal{U}_F = \{F \neq 0\}$. By the argument used in the $\overline{x} \neq 0$ case, $\mathcal{U}_B$ is formally smooth. A similar argument works for $\mathcal{U}_F$: the projection map

$$p : \mathcal{X} \to \operatorname{Spec}\left(\frac{\mathcal{O}[[F, B, C, U, P]]}{(F^2 + 4BC - (\zeta_1 - \zeta_2)^2)} \otimes E\right)$$

is an isomorphism from $\mathcal{U}_F$ onto an open subscheme; but the right hand side is easily seen to be formally smooth. Hence $\mathcal{X}$ is formally smooth. Note that the composition of the map p with the projection away from U is a continuous map with connected fibres and connected image, which admits a continuous section (obtained by taking $R = S = U = 0$); it follows that $\mathcal{X}$ is connected, as required. Since $\mathcal{X}$ is formally smooth it is certainly reduced; therefore $\mathcal{A}/I$ is generically reduced (as it is $\mathcal{O}$-flat), just as we claimed above.

Now we turn to $R^{\square}(\overline{\rho}, \tau_{\zeta,ns})$. By lemma 2.4 we may assume that $\zeta = 1$. The condition that the characteristic polynomial of $\rho^{\square}(\sigma)$ be $(t-1)^2$ is equivalent to the equations:

$$A + D = 0$$
$$AD - (x + B)C = 0.$$

Writing $T = P + Q$ and $U = P - Q$, the condition that

$$q \operatorname{tr}(\rho^{\square}(\phi))^2 = (q+1)^2 \det(\rho^{\square}(\phi))$$

becomes

$$(q-1)^2(T+2)^2 = (q+1)^2(U^2 + 4(y+R)S).$$

Since $t^q - 1 \equiv q(t-1) \mod (t-1)^2$, the Cayley–Hamilton theorem shows that

$$\rho^{\square}(\sigma)^q - 1 = q(\rho^{\square}(\sigma) - 1)$$

on $R^{\square}(\overline{\rho}, \tau_{1,ns})$. From $\phi\sigma\phi^{-1} = \sigma^q$ we therefore get the equation

$$(\phi - 1)(\sigma - 1) - (\sigma - 1)(\phi - 1) = (q-1)(\sigma - 1)\phi$$

on $R^{\square}(\overline{\rho}, \tau_{1,ns})$. Equating coefficients and substituting $D = -A$ we get the equations

$$A^2 + (x+B)C = 0 \tag{21}$$

$$(q-1)^2(T+2)^2 = (q+1)^2(U^2 + 4(y+R)S) \tag{22}$$

$$C(y+R) - S(x+B) = (q-1)(A(1+P) + (x+B)S) \tag{23}$$

$$U(x+B) - 2A(y+R) = (q-1)(A(y+R) + (x+B)(1+Q)) \tag{24}$$

$$2AS - CU = (q-1)(C(1+P) - AS) \tag{25}$$

$$S(x+B) - C(y+R) = (q-1)(C(y+R) - A(1+Q)). \tag{26}$$

After replacing P with $\frac{T+U}{2}$ and Q with $\frac{T-U}{2}$, this is a complete set of equations for $R^{\square}(\overline{\rho}, \tau_{1,ns})$ in $\mathcal{O}[[A,B,C,R,S,T,U]]$.

We replace equations (23) and (26) by their sum and difference:

$$(q-1)(AU + (x+B)S + C(y+R)) = 0 \tag{27}$$

$$(q+1)(C(y+R) - (x+B)S) = (q-1)A(2+T). \tag{28}$$

As $R^{\square}(\overline{\rho}, \tau_{1,ns})$ is λ-torsion free, equation (27) implies that

$$AU + (x+B)S + C(y+R) = 0. \tag{29}$$

We could also write this equation as $\mathrm{tr}((\sigma - 1)\phi) = 0$.

Putting $\alpha(T) = \frac{(q-1)(2+T)}{q+1}$, we find that equations (21), (22), (24),(25) and [(28) and (29)] may respectively be rewritten:

$$\begin{aligned} A^2 + (x+B)C &= 0 \\ 4(y+R)S + (U - \alpha(T))(U + \alpha(T)) &= 0 \\ 2A(y+R) - (x+B)(U - \alpha(T)) &= 0 \\ 2AS - C(U + \alpha(T)) &= 0 \\ 2C(y+R) + A(U - \alpha(T)) &= 0 \\ 2(x+B)S + A(U + \alpha(T)) &= 0. \end{aligned}$$

Let I be the ideal of $\mathcal{O}[[A,B,C,R,S,T,U]]$ generated by these equations and let $R' = \mathcal{O}[[A,B,C,R,S,T,U]]/I$, so that $R^{\square}(\overline{\rho}, \tau_{1,ns})$ is the maximal reduced l-torsion free quotient of R'.

If $\overline{x} \neq 0$ then C, U and S are uniquely determined by A, B, R and T so that

$$R^{\square}(\overline{\rho}, \tau_{1,ns}) \cong \mathcal{O}[[A,B,R,T]].$$

If $\overline{y} \neq 0$, then S, C and A are uniquely determined by B, R, T and U so that

$$R^{\square}(\overline{\rho}, \tau_{1,ns}) \cong \mathcal{O}[[B,R,T,U]].$$

If $\overline{x} = \overline{y} = 0$, so that $x = y = 0$, observe that

$$R' \cong \frac{\mathcal{B}}{J_0 + J_1}$$

where

$$\mathcal{B} = \mathcal{O}[[X_1, \ldots, X_4, Y_1, \ldots, Y_4, T]],$$

the ideal J_0 is generated by the 2×2 minors of

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \end{pmatrix}$$

and $J_1 = (X_1+Y_2, Y_3 - X_4 + 2\frac{q-1}{q+1})$.[1] (The change of variables is $X_1 = A$, $X_2 = B$, $Y_1 = C$, $Y_2 = -A$, $X_3 = -2R/(2+T)$, $Y_4 = 2S(2+T)$, $Y_3 = (U-\alpha(T))/(2+T)$, and $X_4 = (U+\alpha(T))/(2+T)$.) Then by proposition 2.7, $\mathcal{B}/J_0$ is a Cohen–Macaulay, non-Gorenstein domain. Moreover, $(\lambda, X_1+Y_2, X_3-Y_4)$ may be checked to be a regular sequence on $\mathcal{B}/J_0$. Therefore $(X_1+Y_2, X_3+Y_4+2\frac{q-1}{q+1}, \lambda)$ is also regular, and so $\mathcal{B}/(J_0+J_1)$ is Cohen–Macaulay, $\mathcal{O}$-flat and not Gorenstein. The same is then true for R'.

We show that $R' \otimes \mathbb{F}$ is a domain, which implies that R' is a domain. Let $\overline{I}$ be the image of I in $\mathbb{F}[[A,B,C,R,S,T,U]]$. Then $\overline{I}$ is homogeneous so $\mathrm{gr}(R' \otimes \mathbb{F}) = \mathbb{F}[A,B,C,R,S,T,U]/\overline{I}$ and it suffices to check that this is a domain (by [Eis95] corollary 5.5). It is therefore sufficient to check that $\mathrm{Proj}(\mathrm{gr}(R'\otimes\mathbb{F}))$ is reduced and irreducible.[2] But it is easy to check this on the usual seven affine pieces. This argument is from [Tay09].

Next we show that $R^{\square}(\overline{\rho}, \tau_{1,ns})$ is reduced. In fact, we show that

$$\mathcal{Y} = \mathrm{Spec}(R^{\square}(\overline{\rho}, \tau_{1,ns}) \otimes E)$$

is formally smooth, which implies that $R^{\square}(\overline{\rho}, \tau_{1,ns})$ is reduced because it is Cohen–Macaulay and $\mathcal{O}$-flat. For $\star = B$, C, R, S, $U-\alpha(T)$ or $U+\alpha(T)$ let $\mathcal{U}_\star = \{\star \neq 0\} \subset \mathcal{Y}$ be the corresponding affine open subscheme. Then the $\mathcal{U}_\star$ are an affine open cover of $\mathcal{Y}$. For $\star = B$, C, R or S we see that $\mathcal{U}_\star$ is formally smooth by the same argument as for the cases $\overline{x} \neq 0$ and $\overline{y} \neq 0$ above. For $\mathcal{U}_{U\pm\alpha(T)}$, the projection morphism

$$p : \mathcal{U}_{U-\alpha(T)} \to \mathrm{Spec}\left(\frac{\mathcal{O}[[C,R,S,T]]}{4RS - (U+\alpha(T))(U-\alpha(T))} \otimes E\right)$$

is an isomorphism onto an open subscheme. But the right hand scheme is easily seen to be formally smooth as required.

Finally we calculate the $Z^4(\overline{R}(\overline{\rho},\tau))$. We do this when $\overline{x} = \overline{y} = 0$, as the other cases are similar but easier. We have written each $\overline{R}^{\square}(\overline{\rho},\tau)$ as the quotient of $\mathbb{F}[[A,B,C,R,S,T,U]]$ by an ideal which we call $I(\tau)$. Let us recall the presentations:

$$\begin{aligned} I(\tau_{\zeta,s}) &= (A,B,C) \\ I(\tau_{\zeta,ns}) &= (A^2+BC, 4RS+U^2, 2CR+AU, 2BS+AU, 2AR-BU, 2AS-CU) \\ I(\tau_{\zeta_1,\zeta_2}) &= (A^2+BC, BS-CR, 2AR-BU, 2AS-CU) \end{aligned}$$

(using that $A+D=0$ in $\overline{R}^{\square}(\overline{\rho},\tau)$ for each τ, we have eliminated D and written $F = A-D = 2A$). We have already shown that $I(\tau_{\zeta,s})$ and $I(\tau_{\zeta,ns})$ are prime — they are the ideals denoted $\mathfrak{a}_{nr}$ and $\mathfrak{a}_N$ in the statement of the theorem. It is clear that

$$Z^4(\overline{R}^{\square}(\overline{\rho}, \tau_{\zeta,s})) = [\mathfrak{a}_{nr}]$$

and

$$Z^4(\overline{R}^{\square}(\overline{\rho}, \tau_{\zeta,ns})) = [\mathfrak{a}_N].$$

Suppose that $\mathfrak{p}$ is a prime ideal of $\mathbb{F}[[A,B,C,R,S,T,U]]$ containing $I(\tau_{\zeta_1,\zeta_2})$. We show that $\mathfrak{p}$ contains $\mathfrak{a}_{nr}$ or $\mathfrak{a}_N$. If $B, C \in \mathfrak{p}$ then $A \in \mathfrak{p}$ as $A^2+BC \in I(\tau_{\zeta_1,\zeta_2})$,

[1] There is a typo here in the published version.

[2] This argument is not quite correct, see Section 7 for a correction. The error originates with me, not [Tay09].

and we have $\mathfrak{a}_{nr} \subset \mathfrak{p}$. Otherwise, suppose that $B \notin \mathfrak{p}$. As $A^2+BC \in \mathfrak{p}$, either both A and C are in $\mathfrak{p}$ or neither is. If $A, C \in \mathfrak{p}$ then from $2AR - BU \in \mathfrak{p}$ we deduce that $U \in \mathfrak{p}$, while from $BS - CR \in \mathfrak{p}$ we deduce that $S \in \mathfrak{p}$. It is then easy to see that $\mathfrak{a}_N \subset \mathfrak{p}$. If $A, B, C \notin \mathfrak{p}$ then because $B(2CR + AU)$ and $C(2BS + AU)$ are in $I(\tau_{\zeta_1,\zeta_2})$ we see that $2CR+AU, 2BS+AU \in \mathfrak{p}$. This implies that $A(4RS+U^2) \in \mathfrak{p}$, and so $4RS + U^2 \in \mathfrak{p}$ and hence $\mathfrak{a}_N \subset \mathfrak{p}$ as required.

To finish, it is easy to check that

$$e(\overline{R}^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}), \mathfrak{a}_{nr}) = 2$$

and that

$$e(\overline{R}^{\square}(\overline{\rho}, \tau_{\zeta_1,\zeta_2}), \mathfrak{a}_{N}) = 1,$$

and so we get equation 16. □

5.5. **Cohen–Macaulayness.** If τ_0 is a semisimple representation of I_F over E, let $R(\overline{\rho}, \tau_0)'$ be the maximal reduced and l-torsion–free quotient of $R(\overline{\rho})$ all of whose $\overline{E}$-points give rise to representations ρ of G_F with $\rho|_{I_F}^{ss} \cong \tau_0$. Then I claim that $R(\overline{\rho}, \tau_0)'$ is always Cohen–Macaulay. Indeed, if τ_0 is non-scalar then we have proved this above. If τ_0 is scalar, then we may twist and assume that it is trivial. If $q \not\equiv \pm 1 \mod l$, this follows from proposition 5.5. If $q \equiv 1 \mod l$ then we can deduce the claim from proposition 5.8 together with exercise 18.13 of [Eis95], which says that if R/I and R/J are d-dimensional Cohen–Macaulay quotients of a noetherian local ring R, and $\dim R/(I + J) = d - 1$, then $R/(I \cap J)$ is Cohen–Macaulay if and only if $R/(I + J)$ is. We take $R = R^{\square}(\overline{\rho})$, and I and J to be the ideals cutting out $R^{\square}(\overline{\rho}, \tau_s)$ and $R^{\square}(\overline{\rho}, \tau_{ns})$ respectively. Then R/I and R/J are Cohen–Macaulay, and $R/(I+J)$ is a quotient of the formally smooth ring R/I by the single equation $q\,\mathrm{tr}(\rho^{\square}(\phi))^2 = (q+1)^2 \det(\rho^{\square}(\phi))$, and so is Cohen–Macaulay. Therefore $R/(I\cap J)$ is Cohen–Macaulay as required. When $q \equiv -1 \mod l$ the claim follows from proposition 5.6 unless $\overline{\rho}$ is the direct sum of the trivial and cyclotomic characters, in which case we use remark 5.7.

For n-dimensional representations the unrestricted framed deformation ring $R^{\square}(\overline{\rho})$ is always Cohen–Macaulay (in fact, a complete intersection; this is due to David Helm, building on work of Choi [Cho09]). It is natural to wonder whether the rings obtained by fixing the semisimplified restriction to inertia are always Cohen–Macaulay. Note that they are not always Gorenstein.

For a discussion of how the Cohen–Macaulay property of local deformation rings can be used to show that certain global Galois deformation rings are flat over $\mathcal{O}$, see section 5 of [Sno11].

6. Reduction of types – proofs.

The aim of this section is to analyse the reduction modulo l of the K-types $\sigma(\tau)$ defined in section 3, and in particular to prove lemma 3.9.

6.1. **The essentially tame case.** Suppose that $\tau = (r_\tau, N_\tau)$ where r_τ is a tamely ramified, semisimple representation of I_F. Then $\sigma(\tau)$ is inflated from a representation of $GL_2(k_F)$. We will always use the same notation for a representation of $GL_2(k_F)$ and its inflaton to $GL_2(\mathcal{O}_F)$. For this subsection let $G = GL_2(k_F)$, let B be the subgroup of upper-triangular matrices, let U be the subgroup of unipotent elements of B, let Z be the center of G and fix an embedding $\alpha : k_L^\times \hookrightarrow G$. Fix a non-trivial additive character ψ of U. Then we have (see e.g. [BH06] chapter 6):

- If $r_\tau = (\mathrm{rec}(\tilde{\chi}) \oplus \mathrm{rec}(\tilde{\chi}))|_{I_F}$ and $N_\tau \neq 0$, where $\tilde{\chi}|_{\mathcal{O}_F^\times}$ is inflated from a character χ of $k_F^\times$, then
$$\sigma(\tau) = (\chi \circ \det) \otimes \mathrm{St},$$
where St is the Steinberg representation of G;
- If $r_\tau = (\mathrm{rec}(\tilde{\chi}) \oplus \mathrm{rec}(\tilde{\chi}))|_{I_F}$ and $N_\tau = 0$, where $\tilde{\chi}|_{\mathcal{O}_F^\times}$ is inflated from a character χ of $k_F^\times$, then $\sigma(\tau) = \chi \circ \det$;
- If $r_\tau = (\mathrm{rec}(\tilde{\chi}_1) \oplus \mathrm{rec}(\tilde{\chi}_2))|_{I_F}$, where $\tilde{\chi}_1|_{\mathcal{O}_F^\times}$ and $\tilde{\chi}_2|_{\mathcal{O}_F^\times}$ are inflated from *distinct* characters χ_1 and χ_2 of $k_F^\times$, then
$$\sigma(\tau) = \mu(\chi_1, \chi_2)$$
where $\mu(\chi_1, \chi_2) = \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$;
- If $r_\tau = (\mathrm{Ind}_{G_L}^{G_F} \mathrm{rec}(\tilde{\theta}))|_{I_F}$ where $\tilde{\theta}|_{\mathcal{O}_L^\times}$ is inflated from a character θ of $k_L^\times$ which is not equal to its $\mathrm{Gal}(k_L/k_F)$ conjugate θ^c, then
$$\sigma(\tau) = \pi_\theta$$
where $\pi_\theta = \mathrm{Ind}_{ZU}^G(\theta|_Z\psi) - \mathrm{Ind}_{\alpha(k_L^\times)}^G \theta$ (this virtual representation is a genuine irreducible representation that is independent of the choice of ψ).

The only isomorphisms between these representations are of the form $\mu(\chi_1, \chi_2) \cong \mu(\chi_2, \chi_1)$ and $\pi_\theta \cong \pi_{\theta^c}$.

We want to understand the reductions of these representations modulo l, and for this see [Hel10]. We will use analagous notation for representations of G in characteristic zero and in characteristic l; hopefully this will not cause confusion.

If $q \neq \pm 1 \bmod l$, then reduction modulo l is a bijection between irreducible $\overline{\mathbb{F}}_l$-representations of G and irreducible $\overline{E}$-representations of G, as G has order $q(q+1)(q-1)^2$ which is coprime to l.

If $q = 1 \bmod l$, then the distinct irreducible representations of $GL_2(k_F)$ over $\overline{\mathbb{F}}$ are $\chi \circ \det$ and $\mathrm{St} \otimes (\chi \circ \det)$ for $\chi : k_F^\times \to \overline{\mathbb{F}}^\times$, $\mu(\chi_1, \chi_2)$ for $\chi_1, \chi_2 : k_F^\times \to \overline{\mathbb{F}}^\times$ a pair of distinct characters, and π_θ for $\theta : k_L^\times \to \overline{\mathbb{F}}^\times$ character which is not isomorphic to its conjugate. The notation is all entirely analagous to the characteristic zero case. Once again, the only isomorphisms are $\mu(\chi_1, \chi_2) \cong \mu(\chi_2, \chi_1)$ and $\pi_\theta \cong \pi_{\theta^c}$. The reductions of the characteristic zero representations are:

- $\overline{\chi \circ \det} = \overline{\chi} \circ \det$;
- $\overline{\mathrm{St} \otimes \chi \circ \det} = \mathrm{St} \otimes (\overline{\chi} \circ \det)$;
- $\overline{\mu(\chi_1, \chi_2)} = \mu(\overline{\chi}_1, \overline{\chi}_2)$ if $\overline{\chi}_1 \neq \overline{\chi}_2$;
- $\overline{\mu(\chi_1, \chi_2)} = (\overline{\chi} \circ \det) \oplus \mathrm{St} \otimes (\overline{\chi} \circ \det)$ if $\overline{\chi_1} = \overline{\chi_2} = \overline{\chi}$;
- $\overline{\pi}_\theta = \pi_{\overline{\theta}}$.

For the last of these, we must observe that θ/θ^c is a character of $k_L^\times/k_F^\times$, a group which has order $q+1$ and so coprime to l (as $l > 2$). Therefore if $\theta \neq \theta^c$ then $\overline{\theta} \neq \overline{\theta}^c$.

If $q \equiv -1 \bmod l$, then the distinct irreducible representations are: $\chi \circ \det$ for $\chi : k_F^\times \to \overline{\mathbb{F}}^\times$, $\mu(\chi_1, \chi_2)$ for $\chi_1, \chi_2 : k_F^\times \to \overline{\mathbb{F}}^\times$ unordered pair of distinct characters, π_θ for $\theta : k_L^\times \to \overline{\mathbb{F}}^\times$ a character which is not isomorphic to its conjugate, and $(\chi \circ \det) \otimes \pi_1$ for $\chi : k_F^\times \to \overline{\mathbb{F}}^\times$ a character. This last needs some explanation: π_1 is the reduction modulo l of π_θ for any character $\theta : k_L^\times/k_F^\times \to \overline{E}^\times$ which is not equal to θ^c but whose reduction modulo l is trivial. Once again, the only isomorphisms are $\mu(\chi_1,\chi_2) \cong \mu(\chi_2,\chi_1)$ and $\pi_\theta \cong \pi_{\theta^c}$. The reductions of the characteristic 0 representations are:

- $\overline{\chi \circ \det} = \overline{\chi} \circ \det$;
- $\overline{\mu(\chi_1,\chi_2)} = \mu(\overline{\chi}_1,\overline{\chi}_2)$;
- $\overline{\pi}_\theta = \pi_{\overline{\theta}}$ if $\overline{\theta} \neq \overline{\theta}^c$;
- $\overline{\pi}_\theta = \pi_1 \otimes (\overline{\theta}|_{k_F^\times} \circ \det)$ if $\overline{\theta} = \overline{\theta}^c$;
- $\overline{\mathrm{St} \otimes (\chi \circ \det)}$ has $\pi_1 \otimes (\overline{\chi} \circ \det)$ as a submodule with quotient $\overline{\chi} \circ \det$.

In particular, comparing this analysis with lemma 3.8 shows that:

Lemma 6.1. *If $\tau = (r,0)$ and $\tau' = (r',0)$ are scalar on P_F but not on $\tilde{P}_F$, then $\overline{\sigma(\tau)}$ and $\overline{\sigma(\tau')}$ are irreducible and are isomorphic if and only if $r \equiv r' \bmod l$.*

6.2. **The wild case.** If $\tau = (r,0)$ and all twists of r are wildly ramified (we say that τ is 'essentially wildly ramified'), then the following lemma will allow us to show that $\overline{\sigma(\tau)}$ is irreducible. If ρ is a $\overline{\mathbb{Z}}_l$-representation of a group H, we write $\overline{\rho}$ for $\rho \otimes \overline{\mathbb{F}}_l$.

Lemma 6.2. *Suppose that $H \lhd J \subset K$ are profinite groups such that H is open in K, H has pro-order coprime to l, and J/H is an abelian l-group. Suppose that λ is a $\overline{\mathbb{Z}}_l$-representation of J, and write η for the restriction of λ to H. Suppose that η (and hence λ) is irreducible. Suppose that if $g \in K$ intertwines η, then $g \in J$. Then*

1. *The representations of J extending η are precisely $\lambda_i = \lambda \otimes \nu_i$ as ν_i run through the characters of J/H. There is an isomorphism $\mathrm{Ind}_H^J \eta \otimes \overline{E} \cong \bigoplus_i \lambda_i$. The unique $\overline{\mathbb{F}}_l$-representation extending $\overline{\eta}$ is $\overline{\lambda}$, and all of the Jordan–Hölder factors of $\mathrm{Ind}_H^J \overline{\eta}$ are isomorphic to $\overline{\lambda}$.*
2. *A $\overline{\mathbb{F}}_l$-representation ρ of J contains $\overline{\lambda}$ as a subrepresentation if and only if it contains $\overline{\lambda}$ as a quotient.*
3. *The representations $\mathrm{Ind}_J^K \lambda_i$ and $\mathrm{Ind}_J^K \overline{\lambda}$ are irreducible.*

Proof. (1) In characteristic 0 we argue as follows. First note that the representations λ_i are distinct, otherwise $\lambda|_H$ would have a non-scalar endomorphism, contradicting Schur's lemma. By Frobenius reciprocity, the λ_i are distinct irreducible constituents of $\mathrm{Ind}_H^J \eta$. Since the sum of their dimensions is $\dim \mathrm{Ind}_H^J \eta$, they are the only irreducible constituents. By Frobenius reciprocity, any representation extending η must occur in $\mathrm{Ind}_H^J \eta$ and so must be one of the λ_i, as required. In characteristic l, first note that $\overline{\lambda}$ is irreducible since the pro-order of H is coprime to l. It follows from this and the fact that $\overline{\nu}_i$ is trivial for all i that the Jordan–Hölder factors of $\mathrm{Ind}_H^J \overline{\eta}$ are isomorphic to $\overline{\lambda}$. Frobenius reciprocity then implies that $\overline{\lambda}$ is the unique irreducible representation of J extending H.

(2) It follows from part 1 that $\mathrm{Hom}_J(\overline{\lambda},\rho) \neq 0$ if and only if $\mathrm{Hom}_J(\mathrm{Ind}_H^J \overline{\eta},\rho) \neq 0$. By Frobenius reciprocity, this is equivalent to $\mathrm{Hom}_H(\overline{\eta},\rho) \neq 0$. But by the assumption on the pro-order of H, $\overline{\mathbb{F}}_l$-representations of H are semisimple, and so this is equivalent to $\mathrm{Hom}_H(\rho,\overline{\eta}) \neq 0$, which by the same argument is equivalent to $\mathrm{Hom}_J(\rho,\mathrm{Ind}_H^J \overline{\eta}) \neq 0$.

(3) First, note that $\dim \mathrm{Hom}_K(\mathrm{Ind}_J^K \overline{\lambda}, \mathrm{Ind}_J^K \overline{\lambda}) = 1$, by Mackey's decomposition formula and the assumption that elements of $K \setminus J$ do not intertwine η. Now suppose that ρ is an irreducible subrepresentation of $\mathrm{Ind}_J^K \overline{\lambda}$. By

Frobenius reciprocity and part 2 we may deduce that ρ is also an irreducible quotient of $\operatorname{Ind}_J^K \overline{\lambda}$. The composition

$$\operatorname{Ind}_J^K \overline{\lambda} \twoheadrightarrow \rho \hookrightarrow \operatorname{Ind}_J^K \overline{\lambda}$$

is then a non-zero element of $\operatorname{Hom}_K(\operatorname{Ind}_J^K \overline{\lambda}, \operatorname{Ind}_J^K \overline{\lambda})$, and is therefore scalar. But this is only possible if $\rho = \operatorname{Ind}_J^K \overline{\lambda}$, as required. The statement about $\operatorname{Ind}_J^K \lambda_i$ follows. □

Proposition 6.3. *Let $\tau = (r, 0)$ be an essentially wildly ramified inertial type. Then there exists a subgroup $J \subset K$, an irreducible representation λ of J, and a subgroup $\tilde{J} \lhd J$, such that $(\tilde{J}, J, K, \lambda)$ satisfy the hypotheses on (H, J, K, λ) in lemma 6.2 and such that $\sigma(\tau) = \operatorname{Ind}_J^K \lambda$.*

In particular, $\overline{\sigma(\tau)}$ is irreducible.

Proof. Suppose first that r is the restriction to I_F of a reducible representation of G_F. Then $\sigma(\tau) = \operatorname{Ind}_{K_0(N)}^K \epsilon \otimes (\chi \circ \det)$ for a character ϵ of $\mathcal{O}_F^\times$ of exponent $N \geq 2$ and a character χ of $\mathcal{O}_F^\times$. Let $J = K_0(N)$, and let

$$\tilde{J} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in J \ : \ a \text{ has order coprime to } l \text{ modulo } \mathfrak{p}_F \right\}.$$

Then $\tilde{J}$, J and ϵ satisfy all the required hypotheses — the only one to check is that $\epsilon|_{\tilde{J}}$ is not intertwined by any element of $K \setminus J$. We deduce this (in somewhat circular fashion) from the irreducibility of $\operatorname{Ind}_J^K(\epsilon)$, since this is shorter than a direct proof. If $g \in K$ intertwines $\epsilon|_{\tilde{J}}$, then $\operatorname{Hom}_{\tilde{J} \cap g\tilde{J}g^{-1}}(\epsilon, \epsilon^g) \neq 0$. By Mackey's formula,

$$\dim \operatorname{Hom}_{\tilde{J}}(\epsilon, \operatorname{Ind}_{\tilde{J}}^K \epsilon) = \sum_{g \in \tilde{J} \setminus K / \tilde{J}} \dim \operatorname{Hom}_{\tilde{J} \cap g\tilde{J}g^{-1}}(\epsilon, \epsilon^g).$$

The left hand side is in turn equal to $\dim \operatorname{Hom}_K(\operatorname{Ind}_{\tilde{J}}^K \epsilon, \operatorname{Ind}_{\tilde{J}}^K \epsilon)$. But $\operatorname{Ind}_{\tilde{J}}^K \epsilon = \bigoplus_i \operatorname{Ind}_J^K \epsilon_i$ where ϵ_i are the characters of J extending $\epsilon|_{\tilde{J}}$, and by the appendix to [BM02], these $\operatorname{Ind}_J^K \epsilon_i$ are irreducible and distinct. Therefore the left hand side is equal to $(J : \tilde{J})$. The right hand side has a contribution of 1 from each $g \in J/\tilde{J}$, and therefore from no other g, as required.

Now suppose that r is the restriction to I_F of an irreducible representation of G_F. Then $\sigma(\tau) = \operatorname{Ind}_J^K \lambda$ for an irreducible representation λ of J extending an irreducible representation η of a pro-p normal subgroup J^1 of J (see [BH06], sections 15.5, 15.6 and 15.7 — note that our J is the maximal compact subgroup of their J_α, but our J^1 agrees with their J_α^1). We have $J/J^1 = k^\times$, where k is the residue field of a quadratic extension of F, and so J has a normal subgroup $\tilde{J}$ of pro-order coprime to l such that $J/\tilde{J}$ is an l-group. Then $(\tilde{J}, J, K, \lambda)$ satisfy all the required hypotheses — the intertwining statement follows from [BH06], 15.6 Proposition 2. □

Proposition 6.4. *Let $\tau = (r, 0)$ and $\tau' = (r', 0)$ be inertial types that are not scalar on $\tilde{P}_F$. If $r \equiv r' \bmod l$, then $\overline{\sigma(\tau)}$ and $\overline{\sigma(\tau')}$ are isomorphic.*

Proof. If either of r and r' is (after to a twist) tamely ramified, then so is the other and this is contained in lemma 6.1. Otherwise, by lemma 3.8, we are in one of the following cases:

(1) $r = (\chi_1 \oplus \chi_2)|_{I_F}$ for characters χ_1 and χ_2 of G_F that are distinct on P_F, and $r' = (\chi_1' \oplus \chi_2')|_{I_F}$ for characters χ_1' and χ_2' of G_F with $\chi_i \equiv \chi_i'$ for $i = 1, 2$.
(2) $r = (\mathrm{Ind}_{G_L}^{G_F} \xi)_{I_F}$ and $r' = (\mathrm{Ind}_{G_L}^{G_F} \xi')_{I_F}$ for wildly ramified characters ξ and ξ' of G_L such that $\xi \equiv \xi'$, and such that $\xi|_{\tilde{P}_F}$ does not extend to G_F.
(3) $r|_{\tilde{P_F}}$ is irreducible and $r' = r \otimes \chi$ for a character χ of I_F that extends to G_F and such that $\chi \equiv 1 \mod l$.

In the first case, we may write $\chi_i = \mathrm{rec}(\epsilon_i)$ and $\chi_i' = \mathrm{rec}(\epsilon_i')$ with ϵ_i and ϵ_i' characters of $F^\times$ such that $\epsilon_i \equiv \epsilon_i' \mod l$ and such that $\epsilon = \epsilon_1/\epsilon_2$ has exponent $N \geq 1$. Since $\epsilon' = \epsilon_1'/\epsilon_2'$ also has exponent N, we have

$$\begin{aligned}\sigma(\tau) &= \epsilon_2 \otimes \mathrm{Ind}_{K_0(N)}^K \epsilon \\ &\equiv \epsilon_2' \otimes \mathrm{Ind}_{K_0(N)}^K \epsilon' \mod l \\ &= \sigma(\tau').\end{aligned}$$

In the second case, by twisting we may reduce to the case where $(L/F, \mathrm{rec}^{-1}(\xi))$ is an unramified minimal admissible pair ([BH06] paragraph 19.6). Then, following through the explicit construction of [BH06] paragraphs 19.3 and 19.4, we see that there are:

(1) a simple stratum $(\mathfrak{A}, n, \alpha)$ with associated compact open subgroups $J_1 \subset J \subset K$, with J_1 pro-p and $J/J_1 \cong k_L^\times$;
(2) a representation η of J^1 and extensions λ and λ' of η to J such that $\mathrm{Ind}_J^K(\lambda) = \sigma(\tau)$ and $\mathrm{Ind}_J^K(\lambda') = \sigma(\tau')$.

Indeed, up to conjugacy $(\mathfrak{A}, n, \alpha)$, J_1 and η are determined by $\mathrm{rec}^{-1}(\xi)|_{U_L^1} = \mathrm{rec}^{-1}(\xi')|_{U_L^1}$. The representations λ and λ' are defined in terms of $\mathrm{rec}^{-1}(\xi)$ and $\mathrm{rec}^{-1}(\xi')$ by the formulae of [BH06] 19.3.1 and corollary 19.4 (together with the correction factor of paragraph 34.4, an unramified twist Δ_ξ, that makes no difference to the argument). It is clear from these that if $\xi \equiv \xi'$ then $\lambda \equiv \lambda'$ as required.

In the final case, $r' = r \otimes \chi$ for a character χ of I_F that extends to G_F. By compatibility of $\tau \mapsto \sigma(\tau)$ with twisting,

$$\begin{aligned}\sigma(\tau') &= \sigma(\tau) \otimes \mathrm{rec}^{-1}(\chi) \circ \det \\ &\equiv \sigma(\tau) \mod l\end{aligned}$$

as required. □

Proposition 6.5. *Let $\tau = (r, 0)$ and $\tau' = (r', 0)$ be inertial types that are not scalar on $\tilde{P}_F$. If $\overline{\sigma(\tau)}$ and $\overline{\sigma(\tau')}$ are isomorphic, then $r \equiv r' \mod l$.*

Proof. If one of r and r' has a twist which is trivial on P_F, then so does the other and in this case the proposition follows from 6.1.

Otherwise may, by twisting, assume that $\sigma(\tau)$ and $\sigma(\tau')$ satisfy $l(\sigma) \leq l(\sigma \otimes \chi)$ for all characters χ of $\mathcal{O}_F^\times$ (the definition of $l(\sigma)$ is as in [BH06] paragraph 12.6). In this case $\sigma(\tau)$ and $\sigma(\tau')$ contain the same, non-empty, sets of fundamental strata (because this only depends on the restriction to pro-p subgroups).

If one of $\sigma(\tau)$ and $\sigma(\tau')$ contains a split fundamental stratum ([BH06] 13.2) then so does the other. In this case, [BH06] corollary 13.3 implies that they cannot be cuspidal types and so we must have $\sigma(\tau) = \mathrm{Ind}_{K_0(N)}^K(\epsilon)$ and $\sigma(\tau') = \mathrm{Ind}_{K_0(N')}^K(\epsilon')$

for some ϵ and ϵ' of exponents N and N'. It is easy to see that in fact we must have $N = N'$. From lemma 6.2 we deduce that $\epsilon \equiv \epsilon' \mod l$, and so $\tau \equiv \tau' \mod l$ as required.

Otherwise, $\sigma(\tau) = \operatorname{Ind}_J^K \lambda$ and $\sigma(\tau') = \operatorname{Ind}_J^K \lambda'$ for a simple stratum $(\mathfrak{A}, n, \alpha)$ with associated groups $J^1 \subset J$ and representations λ and λ' extending the representation η of J. From lemma 6.2 we deduce that $\lambda' = \lambda \otimes \eta$ for a character η of J/J^1 with $\eta \equiv 1 \mod l$.

If $\mathfrak{A}$ is unramified, then by the reverse of the argument in the second case of the previous proposition we see that $\tau = (\operatorname{Ind}_{G_L}^{G_F} \xi)|_{I_F}$ and $\tau = (\operatorname{Ind}_{G_L}^{G_F} \xi)|_{I_F}$ for ξ and ξ' characters of G_L with $\xi|_{I_L} \equiv \xi'|_{I_L}$, whence the result.

If $\mathfrak{A}$ is ramified, then η can be regarded as a character of $J/J^1 \cong k_M^\times = k_F^\times$ with $\eta \equiv 1 \mod l$ for some *ramified* quadratic extension M/F. I claim that there is a character χ of $\mathcal{O}_F^\times$ with $\eta = \chi \circ \det$ and $\chi \equiv 1 \mod l$. Indeed, as $l > 2$ we can take the inflation to $\mathcal{O}_F^\times$ of the character χ of $k_F^\times$ satisfying $\chi \equiv 1 \mod l$ and $\chi^2 = \eta$. Then $\sigma(\tau) = \sigma(\tau') \otimes (\chi \circ \det)$ and so

$$\begin{aligned} \tau &= \tau' \otimes \operatorname{rec}(\chi) \\ &\equiv \tau' \mod l \end{aligned}$$

as required. □

7. Erratum

The proof of Proposition 2.7 is not correct; however, the proposition is true and the results of the paper are unaffected. There is a related gap in the proof of Proposition 5.8, which we also fill. I am very grateful to Lue Pan for pointing out the error.

The problem is that $\operatorname{Proj}(S/(X_1 - \alpha_1 Y_1, \dots, X_i - \alpha_i Y_i))$ being reduced doesn't imply that $S/(X_1 - \alpha_1 Y_1, \dots, X_i - \alpha_i Y_i)$ is reduced — there may be nilpotent elements annihilated by the 'irrelevant ideal' generated by positively graded elements.

However, the given reference ([Eis95] Theorem 18.18) certainly implies that S/I is Cohen–Macaulay; it follows that the claimed sequence is in fact a regular sequence, and the characterisation of when S/I is Gorenstein follows as in the given proof.

Alternatively we can use an argument that I learned from the MathOverflow posts [hco] and [ah]. It is well-known that R is the homogeneous coordinate ring of the image X of the Segre embedding of $s : \mathbb{P}^1 \times \mathbb{P}^{n-1} \to \mathbb{P}^{2n-1}$. Then R is Cohen–Macaulay if and only if $H^i(\mathbb{P}^{2n-1}, \mathcal{I}_X(r)) = 0$ for all $0 < i < n$ and all $r \in \mathbb{Z}$, and R is Gorenstein if, in addition, $\omega_X \cong \mathcal{O}_{\mathbb{P}^{2n-1}}(r)|_X$ for some $r \in \mathbb{Z}$ (see [Mig98, pp9-11, proposition 4.1.1]). From the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^{2n-1}} \to \mathcal{O}_X,$$

the equation $s^*\mathcal{O}_{\mathbb{P}^{2n-1}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and the Künneth formula we see that R is Cohen–Macaulay. Since $\omega_X \cong \mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(-n)$, we see that R is Gorenstein if and only if $n = 2$.

A similar issue affects the proof of Proposition 5.8, in the sentence "It is therefore sufficient to check that $\operatorname{Proj}(\operatorname{gr}(R' \otimes \mathbb{F}))$ is reduced and irreducible.". It is not. However, the given argument shows that $\operatorname{gr}(R' \otimes \mathbb{F})$ has a unique minimal prime ideal and that any nilpotent elements are supported at the irrelevant ideal. But we know that this ring is Cohen–Macaulay and so has no embedded associated primes.

It follows that $\mathrm{gr}(R' \otimes \mathbb{F})$ is reduced with a unique minimal prime ideal, and is therefore a domain as claimed.

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