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DEGENERATE WHITTAKER FUNCTIONS FOR $\mathrm{Sp}_n(\mathbb{R})$

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Abstract. In this paper, we construct Whittaker functions with exponential growth for the degenerate principal series of the symplectic group of genus n induced from the Siegel parabolic subgroup. This is achieved by explicitly constructing a certain Goodman-Wallach operator which yields an intertwining map from the degenerate principal series to the space of Whittaker functions, and by evaluating it on weight ℓ standard sections. We define a differential operator on such Whittaker functions which can be viewed as generalization of the ξ-operator on harmonic Maass forms for $\mathrm{SL}_2(\mathbb{R})$.

1. Introduction

The standard theory of automorphic forms focuses on the spectral decomposition of the space $L^2(\Gamma\backslash G)$ where G is a connected real semi-simple Lie group and Γ is a discrete subgroup of finite co-volume. This analysis involves functions of rapid decay, e.g, cusp forms, or of moderate growth, e.g., Eisenstein series. Classically, in the case of $G = \mathrm{SL}_2(\mathbb{R})$ and Γ commensurable to $\mathrm{SL}_2(\mathbb{Z})$, the Fourier expansions of such functions involves the solution of the Whittaker ordinary differential equation that decays exponentially at infinity; this solution is uniquely characterized by this decay. In general, the uniqueness of the Whittaker model and the associated smooth Whittaker functional, Jacquet's functional, plays a fundamental role and is the subject of a vast literature.

Other Whittaker functionals and the associated 'bad' Whittaker functions, which grow exponentially at infinity, have played a less prominent role in the theory of automorphic forms. An important exception to this is the work of Miatello and Wallach, [21], where, for G semi-simple of split rank 1, a theory of Poincaré series constructed from such Whittaker functions is developed. More recently, in the case of $G = \mathrm{SL}_2(\mathbb{R})$ and Γ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the space of (vector-valued) weak Maass forms – Maass forms that are allowed to grow exponentially at the cusps – and its subspace of harmonic weak Maass forms, analytic functions annihilated by the weight k Laplacian, have been shown to have interesting and important arithmetic applications, [8], [9], [11]. For example, the weakly holomorphic modular forms are the input for Borcherds celebrated construction of meromorphic modular forms with product formulas, [3], and, more generally, harmonic Maass forms can be used to construct Arakelov type Green functions for special divisors on orthogonal and unitary Shimura varieties, [7], [9]. The harmonic weak Maass forms of negative (or low) weight play a role in the theory of mock modular forms and their relatives, [5], [29], [28]. In particular,

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they are linked to holomorphic modular forms of a complementary positive weight by means of the ξ-operator introduced in [7].

There are serious obstacles to extending these results to groups of higher rank. For example, the Koecher principle asserts that, for Γ irreducible in G of hermitian type of reduced rank greater than 1, any holomorphic modular form on the associated bounded symmetric domain 'extends holomorphically' to the cusps, i.e., the notions of holomorphic modular form and weakly holomorphic modular form coincide. More generally, Miatello and Wallach conjecture, [21], Section 5, that the same phenomenon occurs for general automorphic forms. Specifically, they conjecture that, for Γ irreducible in G of real reduced rank greater than 1, a smooth function f on $\Gamma\backslash G$ that is K-finite and an eigenfunction of the center of the universal enveloping algebra of $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ is automatically of moderate growth. They prove this conjecture in the case of $SO(n,1)$ over a totally real field.

However, in [6], the first author has shown that for $G = \mathrm{SL}_2(\mathbb{R})^d$ and Γ an arithmetic subgroup Γ of $\mathrm{SL}_2(O_{\boldsymbol{k}})$, where $O_{\boldsymbol{k}}$ is the ring of integers in a totally real field $\boldsymbol{k}$ with $|\boldsymbol{k}:\mathbb{Q}| = d > 1$, it is possible to replace the non-existent space of harmonic weak Maass forms with a certain space of Whittaker functions. These Whittaker functions are invariant only under the unipotent subgroup Γ_∞^u of Γ_∞ and the associated Poincaré series do not converge. Nevertheless, it is shown in [6] that they are linked to holomorphic Hilbert modular cusps forms via a ξ-operator and provide an adequate input for a Borcherds type construction. This suggests that it would be fruitful to consider analogous Whittaker functions for more general groups.

The goal of this paper is to construct 'bad' Whittaker functions for the degenerate principal series $I(s)$ induced from a character of the Levi factor $M = \mathrm{GL}_n(\mathbb{R})$ of the Siegel parabolic $P = MN$ of $G = \mathrm{Sp}_n(\mathbb{R})$. For $s \in \mathbb{C}$, let $I(s) = I(s,\chi)$ be the space of smooth of K-finite functions ϕ on G such that

$$\phi(n(b)m(a)g) = \chi(\det a)|\det(a)|^{s+\rho}\phi(g), \quad m(a) = \begin{pmatrix} a & \\ & {}^ta^{-1}\end{pmatrix},\ n(b) = \begin{pmatrix} 1 & b\\ & 1\end{pmatrix}, \tag{1.1}$$

$a \in \mathrm{GL}_n(\mathbb{R})$, $b \in S := \mathrm{Sym}_n(\mathbb{R})$, $\chi(t) = \mathrm{sgn}(t)^\nu$, $\nu = 0, 1$, $\rho = \frac{1}{2}(n+1)$. Then $I(s)$ is a $(\mathfrak{g}, K)$-module, where $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ and $K \simeq U(n)$, $k \mapsto \mathbf{k}$. For $T \in \mathrm{Sym}_n(\mathbb{R})$ with $\det(T) \neq 0$, an algebraic Whittaker functional of type T is an element $\boldsymbol{\omega}^T \in I(s)^* = \mathrm{Hom}_{\mathbb{C}}(I(s),\mathbb{C})$ such that

$$\boldsymbol{\omega}^T(\mathfrak{n}(X)\phi) = 2\pi i\,\mathrm{tr}(TX)\,\boldsymbol{\omega}^T(\phi), \quad \mathfrak{n}(X) = \begin{pmatrix} & X\\ 0 & \end{pmatrix} \in \mathfrak{g}, \quad X \in S_{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C}). \tag{1.2}$$

Such a functional determines a $(\mathfrak{g}, K)$-intertwining map

$$\omega^T : I(s) \longrightarrow \mathcal{W}^T(G), \qquad \omega^T(\phi)(g) = \boldsymbol{\omega}^T(\pi(g)\phi), \tag{1.3}$$

where $\mathcal{W}^T(G)$ is the space of smooth functions f on G such that

$$f(n(b)g)) = e(\mathrm{tr}(Tb))\,f(g). \tag{1.4}$$

Here $e(t) = e^{2\pi i t}$. The resulting generalized Whittaker functions are right K-finite and real analytic on G. Conversely, such an intertwining map (1.3) gives rise to a Whittaker functional $\boldsymbol{\omega}^T(\phi) = \omega^T(\phi)(e)$.

One intertwining map is given by the integral

$$W^T(g,s;\phi) = \int_S \phi(\underline{w}n(b)g)\, e(-\mathrm{tr}(Tb))\, db, \qquad \underline{w} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

which converges absolutely for $\mathrm{Re}(s) > \rho$ and has a meromorphic analytic continuation in s. For example, for $\phi = \phi_{s,\ell}$, the (unique) function in $I(s)$ such that $\phi_{s,\ell}(k) = \det(\mathbf{k})^\ell$,

$$W^T(n(b)m(a)k, s; \phi_{s,\ell}) = \chi(\det(a))\,|\det(a)|^{s+\rho}\, e(\mathrm{tr}(Tb))\, \det(\mathbf{k})^\ell\, \xi(v,T;\alpha',\beta'), \tag{1.5}$$

where

$$\xi(v,T;\alpha',\beta') = \int_S \det(b+iv)^{-\alpha'}\, \det(b-iv)^{-\beta'}\, e(-\mathrm{tr}(Tb))\, db, \tag{1.6}$$

with $v = a\,{}^t a$, $\alpha' = \frac{1}{2}(s+\rho+\ell)$ and $\beta' = \frac{1}{2}(s+\rho-\ell)$, is the confluent hypergeometric function of matrix argument studied by Shimura [23]. If $\epsilon T > 0$ with $\epsilon = \pm 1$, then (1.6) can be written as

$$\begin{aligned} &i^{-n\ell}\, 2^{-n(\rho-1)}\, (2\pi)^{n(s+\rho)} \Gamma_n(\alpha)^{-1}\Gamma_n(\beta)^{-1}\, |\det(T)|^s \\ &\quad\times\, e^{-2\pi\epsilon \mathrm{tr}(Tv)} \int_{t>0} e^{-2\pi \mathrm{tr}(cv\,{}^t c t)}\, \det(t)^{\alpha-\rho}\, \det(t+1)^{\beta-\rho}\, dt, \end{aligned} \tag{1.7}$$

where $\alpha = \frac{1}{2}(s+\rho-\epsilon\ell)$ and $\beta = \frac{1}{2}(s+\rho+\epsilon\ell)$ and $\epsilon T = c\,{}^t c$. For notation not explained here see section 2.1 and the Appendix.

The Whittaker function (1.5), which decays exponentially as the trace of v goes to infinity, plays a key role in many applications. The corresponding Whittaker functional on $I(s)$ is characterized, among all algebraic Whittaker functionals, by the fact that it extends to a continuous functional on the space $I^{\mathrm{sm}}(s)$ of smooth functions on G satisfying (1.1). In the general theory, such Jacquet functionals and the resulting good Whittaker functions have been studied very extensively, cf. [24] and the literature discussed there.

To construct other Whittaker functionals, we apply Goodman-Wallach operators to conical vectors in $I(s)^*$. Such conical vectors correspond to embeddings into induced representations. The two relevant ones in our situation are given by

$$c_1(\phi) = \phi(e), \tag{1.8}$$

and, for $\mathrm{Re}(s) > \rho$,

$$c_{\underline{w}}(\phi) = (A(s,\underline{w})\phi)(e) = \int_S \phi(\underline{w}n(b))\, db, \tag{1.9}$$

where $A(s,\underline{w}) : I(s) \longrightarrow I(-s)$ is the intertwining operator defined by (3.3), with corresponding embeddings the identity map and $A(s,\underline{w})$ respectively. Matumoto's generalization [20] of the results of [13] apply in our situation. Let

$$\bar{N} = \{\, n_-(x) \mid x \in S \,\}, \qquad n_-(x) = \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}, \tag{1.10}$$

and let $\bar{\mathfrak{n}} = \mathrm{Lie}(\bar{N})_{\mathbb{C}}$. Since $\bar{\mathfrak{n}}$ is abelian, $U(\bar{\mathfrak{n}}) = S(\bar{\mathfrak{n}})$, and the completion

$$S(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]} = \varprojlim_r S(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^r S(\bar{\mathfrak{n}}) \tag{1.11}$$

is the ring of formal power series in elements of $\bar{\mathfrak{n}}$. The action of $U(\bar{\mathfrak{n}})$ on $I(s)^*$ extends[1] to an action of $S(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]}$. Then, by the results of [20], there are elements

$$\mathrm{gw}_s^T, \quad \mathrm{gw}_{-s}^T \in S(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]}$$

such that

$$\boldsymbol{\omega}_1^T := \mathrm{gw}_s^T \cdot c_1, \quad \boldsymbol{\omega}_{\underline{w}}^T := \mathrm{gw}_{-s}^T \cdot c_{\underline{w}}$$

are Whittaker functionals of type T.

Following [13] one realizes the representation $I(s)$ on a space of functions on $\bar{N}$, where the Goodman-Wallach operator is given by a differential operator of infinite order. Taking advantage of the fact that N is abelian and passing to the Fourier transform, this operator is realized as multiplication by an analytic function. In the case of $\mathrm{SL}_2(\mathbb{R})$, this function is given explicitly in the introduction to [13], where the corresponding formal power series in $S(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]}$ is determined by a simple recursion relation. For $n \geq 2$, it does not seem feasible to apply this method, and, as far as we could see, explicit formulas for these kernel functions do not exist in the literature.

Our first main result is that, in analogy to the case of $\mathrm{SL}_2(\mathbb{R})$ given in [13], the kernel function for the Goodman-Wallach operator gw_s^T for any n is given explicitly by a Bessel function, now of a matrix argument. We now describe the result in more detail. For $\phi \in I(s)$, define a function $\Psi(\phi)$ in $\bar{N}$ by

$$\Psi(x;\phi) = \phi(n_-(x)) \tag{1.12}$$

and let

$$\hat{\Psi}(y;\phi) = \int_S e(\mathrm{tr}(xy))\, \Psi(x;\phi)\, dx \tag{1.13}$$

be its Fourier transform. Note that, in this model of $I(s)$, the conical functional c_1 is given by

$$c_1(\phi) = \Psi(0;\phi) = \int_S \hat{\Psi}(y;\phi)\, dy. \tag{1.14}$$

To define the relevant hypergeometric function of matrix argument, we use the notation and results of [12] and [22]. For z and $w \in S_{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C})$ and for $\mathbf{m} = (m_1, \dots, m_n)$, with integers m_j with $m_1 \geq m_2 \geq \cdots \geq m_n$, let $\Phi_{\mathbf{m}}(z)$ be the spherical polynomial, and let $\Phi_{\mathbf{m}}(z,w)$ be its 'bi-variant' version. For further explanation and notation, see Appendix 1. Following [22], define the hypergeometric function

$$\mathrm{GW}_s(z,w) := \sum_{\mathbf{m}\geq 0} \frac{(-1)^{|\mathbf{m}|}\, d_{\mathbf{m}}}{(s+\rho)_{\mathbf{m}}(\rho)_{\mathbf{m}}}\, \Phi_{\mathbf{m}}(z,w). \tag{1.15}$$

This is a Bessel function of matrix argument, [14], [12].

Theorem A. *The kernel for the Goodman-Wallach operator is given by* $\mathrm{GW}_s(\cdot, 2\pi T)$. *More precisely, for* $\phi \in I(s)$,

$$\boldsymbol{\omega}_1^T(\phi) = \int_S \mathrm{GW}_s(2\pi y, 2\pi T)\, \hat{\Psi}(y;\phi)\, dy.$$

[1]Here it is essential that we have taken the dual $I(s)^*$ of the K-finite vectors $I(s)$.

Thus, the corresponding Whittaker function is

$$\omega_1^T(g;\phi) = \boldsymbol{\omega}_1^T(\pi(g)\phi) = \int_S \mathrm{GW}_s(2\pi y, 2\pi T)\, \hat{\Psi}(y;\pi(g)\phi)\, dy.$$

Our second main result is an evaluation of the Whittaker function for $\phi = \phi_{s,\ell}$.

Theorem B. *For $\epsilon = \pm 1$, suppose that $\epsilon T \in \mathrm{Sym}_n(\mathbb{R})_{>0}$ and let $f^T_{s,\ell}(g) = \omega_1^T(\pi(g)\phi_{s,\ell})$ be the weight ℓ degenerate Whittaker function. Then, for $g = n(b)m(a)k$,*

$$f^T_{s,\ell}(g) = \chi(\det(a))\,|\det(a)|^{s+\rho}\, e(\mathrm{tr}(Tb))\, \det(\mathbf{k})^\ell \\ \times 2^{n(s-\rho+1)}\, \exp(-2\pi\,\epsilon\,\mathrm{tr}(Tv))\, {}_1F_1(\alpha,\alpha+\beta; 4\pi c v\, {}^t c)$$

where $v = a\,{}^t a$, $\alpha = \frac{1}{2}(s+\rho-\epsilon\ell)$ and $\beta = \frac{1}{2}(s+\rho+\epsilon\ell)$ and $\epsilon T = {}^t c c$. Here

$$\begin{aligned} {}_1F_1(\alpha,\alpha+\beta;z) &= \sum_{\mathbf{m}\ge 0} \frac{(\alpha)_{\mathbf{m}}}{(\alpha+\beta)_{\mathbf{m}}} \frac{d_{\mathbf{m}}}{(\rho)_{\mathbf{m}}}\, \Phi_{\mathbf{m}}(z) \\ &= \frac{\Gamma_n(\alpha+\beta)}{\Gamma_n(\alpha)\Gamma_n(\beta)} \int\limits_{\substack{t>0\\ 1-t>0}} e^{\mathrm{tr}(zt)}\, \det(t)^{\alpha-\rho}\, \det(1-t)^{\beta-\rho}\, dt \end{aligned}$$

is the matrix argument hypergeometric function.

It is instructive to compare the formula for $f^T_{s,\ell}$ and the integral occurring here with the expressions in (1.5) and (1.7) defining the good Whittaker function. Note, for example, that in the case $n = 1$, the functions

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b)} \int_0^1 e^{zt}\, t^{a-1}\, (1-t)^{b-a-1}\, dt$$

and

$$U(a,b;z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt}\, t^{a-1}\, (t+1)^{b-a-1}\, dt,$$

occurring in Theorem A and (1.7) respectively, are a standard basis for the space of solutions for the second order Kummer equation, [1], Chapter 13.

The proof of Theorem B depends on an elaborate calculation which makes essential use of the fact that, up to diagonalization, the orthogonal group of T is $O(n)$. Thus, at present, we do not have a corresponding evaluation for T of arbitrary signature.

It is easy to check, cf. Lemma 7.1, that for a, b, and $z > 0$ real, with $a > \rho$, $b > \rho$, and for any η with $0 < \eta < 1$,

$$|{}_1F_1(a, a+b; z)| \ge C_\eta\, e^{(1-\eta)\mathrm{tr}(z)}.$$

Thus, for s real and α and $\beta > \rho$,

$$|f^T_{s,\ell}(g)| \ge C'_\eta\, \det(v)^{\frac{1}{2}(s+\rho)}\, e^{2\pi(1-2\eta)\,\mathrm{tr}(\epsilon T v)}.$$

This shows the exponential growth of $f^T_{s,\ell}$ as the trace of v goes to infinity.

Note that, due to its construction via a Whittaker functional, the function $f^T_{s,\ell}$ on G is an eigenfunction for the center of the universal enveloping algebra with eigencharacter given by the infinitesimal character of the degenerate principal series $I(s)$.

Of course, for $n = 1$, the results of Theorems A and B agree with the expressions given in the introduction of [13] in the case of $\mathrm{SL}_2(\mathbb{R})$. Also, up to an elementary factor, the Whittaker function

$$f^T_{s,\ell}(g) = \chi(a)\,|a|^{s+\rho}\,e(Tb)\,\det(\mathbf{k})^\ell\,2^s\,\exp(-2\pi\,\epsilon\,Tv)\,M(\alpha,\alpha+\beta;4\pi\epsilon Tv)$$

in the $n = 1$ case is precisely (the m_1-component of) the function utilized in the construction of [6], (4.13).

Finally, we define an analogue of the ξ-operator introduced in [7] and [6]. Since this operator is a variant of the $\bar\partial$-operator, it is best expressed in terms of vector bundles. Write $\mathfrak{H}_n$ for the Siegel upper half plane of genus n. For a discrete, torsion free subgroup $\Gamma_0 \subset G$, let $X = \Gamma_0\backslash\mathfrak{H}_n$, and let $\mathcal{E}^{a,b}$ be the bundle of smooth differential forms of type (a, b) on X. For a hermitian vector bundle E on X, let

$$\bar{*}_E : \mathcal{E}^{a,b} \otimes E \longrightarrow \mathcal{E}^{N-a,N-b} \otimes E^*$$

be the Hodge $*$-operator, [27], Chapter V, Section 2. Here $N = n\rho = \dim \mathfrak{H}_n$. Note that we include the cases where Γ_0 is Γ^u_∞ or trivial.

Definition. For a hermitian vector bundle E on X, the ξ-operator is defined as

$$\xi = \xi_E = \bar{*}_E \circ \bar\partial : \Gamma(X, \mathcal{E}^{a,b} \otimes E) \;\longrightarrow\; \Gamma(X, \mathcal{E}^{N-a,N-b-1} \otimes E^*). \tag{1.16}$$

For a finite dimensional representation $(\sigma, \mathcal{V}_\sigma)$ of $\mathrm{GL}_n(\mathbb{C})$ with an admissible hermitian norm, there is an associated homogeneous hermitian vector bundle $\mathcal{L}_\sigma$ on X. For example, for an integer r, $\mathcal{L}_r := \mathcal{L}_{(\det)^{-r}}$ is the line bundle whose sections correspond to functions on $\mathfrak{H}_n$ that transform like Siegel modular forms of weight r with respect to Γ_0. In particular, $\mathcal{E}^{N,0} \simeq \mathcal{L}_{n+1}$. If $\mathcal{F}_\nu$ is a flat hermitian bundle associated to a unitary representation $(\nu, \mathcal{F}_\nu)$ of Γ_0, and κ is an integer, then sections of $\mathcal{F}_\nu \otimes \mathcal{E}^{0,N-1} \otimes \mathcal{L}_{n+1-\kappa}$ can be viewed as $\mathcal{F}_\nu$-valued $(0, N-1)$-forms of weight $n+1-\kappa$, and ξ carries such sections to sections of

$$\mathcal{F}_{\nu^\vee} \otimes \mathcal{E}^{N,0} \otimes \mathcal{L}_{\kappa-n-1} \simeq \mathcal{F}_{\nu^\vee} \otimes \mathcal{L}_\kappa.$$

For $n = 1$ and Γ_0 a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$, this reduces to the ξ-operator defined in [7] where $(\nu, \mathcal{F}_\nu)$ is a finite Weil representation. For simplicity, we now omit the bundle $\mathcal{F}_\nu$.

Motivated by the construction of [6], we consider the ξ-operator applied to a space of Whittaker forms

$$\xi :\; \mathcal{W}^{-T}(\mathcal{E}^{0,N-1} \otimes \mathcal{L}_{n+1-\kappa}) \longrightarrow \mathcal{W}^T(\mathcal{L}_\kappa).$$

Here we take $T \in \mathrm{Sym}_n(\mathbb{Z})^\vee_{>0}$ and $\Gamma_0 = \Gamma^u_\infty = \mathrm{Sp}_n(\mathbb{Z}) \cap N$. Lifted to G, this amounts to

$$\xi :\; [\,\mathcal{W}^{-T}(G) \otimes \wedge^{N-1}(\mathfrak{p}^*_-) \otimes \mathbb{C}(\kappa-n-1)\,]^K \longrightarrow [\,\mathcal{W}^T(G) \otimes \mathbb{C}(-\kappa)\,]^K.$$

A family of Whittaker forms in the space on the left here can be constructed by means of our Whittaker functional. Note that

$$\sigma^\vee = \wedge^{N-1}(\mathfrak{p}_-^*) \otimes \mathbb{C}(\kappa - n - 1)$$

is an irreducible representation of K. Since the K-types of $I(s)$ occur with multiplicity one, we see that the space

$$[\ I(s) \otimes \wedge^{N-1}(\mathfrak{p}_-^*) \otimes \mathbb{C}(\kappa - n - 1)\ ]^K$$

has dimension 1. Let $\phi_{s,\sigma}$ be a basis vector. We then obtain a diagram

(1.17)

$$\begin{array}{ccc} \big[\, I(s) \otimes \wedge^{N-1}(\mathfrak{p}_-^*) \otimes \mathbb{C}(\kappa - n - 1)\ \big]^K & \xrightarrow{\omega_1^{-T}} & \big[\, \mathcal{W}^{-T}(G) \otimes \wedge^{N-1}(\mathfrak{p}_-^*) \otimes \mathbb{C}(\kappa - n - 1)\ \big]^K \\ \downarrow \xi & & \downarrow \xi \\ \big[\, I(\bar{s}) \otimes \mathbb{C}(-\kappa)\ \big]^K & \xrightarrow{\omega_1^{T}} & \big[\, \mathcal{W}^{T}(G) \otimes \mathbb{C}(-\kappa)\ \big]^K. \end{array}$$

and define the Whittaker form

$$\boldsymbol{f}_{s,\sigma}^{-T} := \omega_1^{-T}(\phi_{s,\sigma}).$$

We finally determine the behavior of this family of Whittaker forms under the ξ-operator.

Theorem C. *The Whittaker form $\boldsymbol{f}_{s,\sigma}^{-T}$ has the following properties.*
(i) The form $\boldsymbol{f}_{s,\sigma}^{-T}$ is an eigenfunction of the center of the universal enveloping algebra of $\mathfrak{g}$ with eigencharacter the infinitesimal character of $I(s)$. In particular, for the Casimir operator C,

$$C \cdot \boldsymbol{f}_{s,\sigma}^{-T} = \frac{1}{8}\,(s+\rho)(s-\rho)\,\boldsymbol{f}_{s,\sigma}^{-T}.$$

(ii) For the ξ-operator,

$$\xi(\boldsymbol{f}_{s,\sigma}^{-T}) = n\,(\bar{s} - \rho + \kappa)\,\overline{f_{s,-\kappa}^{-T}},$$

where

$$f_{s,-\kappa}^{-T} = \omega_1^{-T}(\phi_{s,-\kappa}).$$

(iii) At $s = s_0 = \kappa - \rho$,

$$\xi(\boldsymbol{f}_{s_0,\sigma}^{-T})(g) = c(n, s_0)\, W_\kappa^T(g),$$

where

$$W_\kappa^T(n(b)m(a)k) = \det(\mathbf{k})^\kappa\, \det(a)^\kappa\, e(\mathrm{tr}(T\tau)) = j(g,i)^{-\kappa}\, q^T, \tag{1.18}$$

and $c(n, s_0) = 2^{n(\kappa - 2\rho + 1)}$. Here $\tau = b + ia{}^ta$ and $q^T = e(\mathrm{tr}(T\tau))$.

Thus, the ξ-operator carries $\boldsymbol{f}_{s_0,\sigma}^{-T}$ to the standard holomorphic Whittaker function of weight κ.

Here note that, for $\Gamma = \mathrm{Sp}_n(\mathbb{Z})$ and for $\kappa > 2n$, the Poincaré series defined by

$$\mathcal{P}_\Gamma(W_\kappa^T)(g) = \sum_{\gamma \in \Gamma_\infty^u \backslash \Gamma} W_\kappa^T(\gamma g)$$

is termwise absolutely convergent and defines a cusp form of weight κ. We define the global ξ-operator

$$\xi_\Gamma = \mathcal{P}_\Gamma \circ \xi \ : \mathbb{H}_{n+1-\kappa}(G) \longrightarrow S_\kappa(\Gamma),$$

where $\mathbb{H}_{n+1-\kappa}$ is the subspace of

$$[\ C^\infty(\Gamma_\infty^u \backslash G) \otimes \wedge^{N-1}(\mathfrak{p}_-^*)\ ]^K$$

spanned by the $\boldsymbol{f}_{s_0,\sigma}^{-T}$ for $T \in \mathrm{Sym}_n(\mathbb{Z})_{>0}^\vee$. Since the Poincaré series span $S_\kappa(\Gamma)$, the conjugate linear map ξ_Γ is surjective and we obtain a kind of 'resolution'

$$\ker(\xi_\Gamma) \longrightarrow \mathbb{H}_{n+1-\kappa} \longrightarrow S_\kappa(\Gamma) \tag{1.19}$$

of the space of cusp forms which, for $n > 1$, might be viewed as a kind of replacement for harmonic weak Maass forms in the higher genus case.

It is our hope that the 'resolution' of the space of cusp forms resulting from (1.19) will have interesting arithmetic applications. In particular, we will consider the Borcherds lift/regularized theta lift of such forms in a sequel to this paper.

We remark that there are two points where our results could be extended. First, we have only determined the Whittaker function $\omega_1^T(\phi_{s,\ell})$ for definite T. As mentioned above and explained in Section 4, our calculation depends on this assumption in an essential way, although it may be that some variant could be used for T of arbitrary signature. Note that this case distinction also occurs in [23] where the case of arbitrary signature requires a more elaborate argument. Second, we have not treated the Whittaker functions $\boldsymbol{\omega}_{\underline{w}}^T(f_{s,\ell})$ arising from the other conical vector $c_{\underline{w}}$. There are two reasons for this. On the one hand, we do not need them for the applications we have in mind, and, on the other hand, already in the case $n = 1$, some additional complications arise which we did not see how to handle for general n.

We now briefly describe the contents of the various sections. In Section 2, we review background material about the degenerate principal series representation $I(s)$. In Section 3, we begin with a sketch of the theory of Goodman-Wallach operators relevant to our situation, intended to summarize some of the basic ideas from [13] and [20] for nonspecialists (like the authors). We then state and prove our first main result, Theorem 3.1 (Theorem A). Its proof depends on some basic facts about matrix argument Bessel functions and Bessel operators from [22]. Note that everything up to this point could just as well have been formulated in terms of analysis on symmetric cones associated to Euclidean Jordan algebras, as in [12], [22], [23], and it should be possible to prove the analogue of Theorem A in this generality. We plan to do this in a sequel. In Section 4, we compute that 'bad' Whittaker function with scalar K-type explicitly via an elaborate exercise with special functions of matrix argument. As the final answer is quite simple, we wonder if there is not a more direct derivation of it but did not succeed in finding one. In Section 5, we begin by defining the ξ-operator in some generality. We then show that its action on Whittaker forms can be determined from that of the corresponding operator on a complex associated to the degenerate principal series, cf. (5.16). We then construct certain Whittaker forms whose images under the ξ-operator interpolate, in the variable s, the standard Whittaker function W_κ^T occurring in the Fourier expansion of holomorphic Siegel cusp forms of weight κ, as explained in Theorem C. In Section 6, we briefly discuss the global ξ-operator and in Section 7, the Appendix, we review some notation from [12], the inversion formula used in the proof of Theorem A, and an estimate for the growth of ${}_1F_1$.

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2. Background

2.1. **Notation.** Let W, $\langle\,,\,\rangle$ be a symplectic vector space of dimension $2n$ over $\mathbb{Q}$ with standard basis $e_1,\dots,e_n,f_1,\dots,f_n$ with $\langle\, e_i, f_j\,\rangle = \delta_{ij}$ and $\langle\, e_i, e_j\,\rangle = \langle\, f_i, f_j\,\rangle = 0$. Let $G = \mathrm{Sp}(W) \simeq \mathrm{Sp}_n/\mathbb{Q}$. Following the tradition of [25] and [26], we view W as a space of row vectors with G acting on the right. The Siegel parabolic P is the stabilizer of the subspace spanned by the f_j's, and we write $P = MN$ with Levi subgroup

$$M = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_n \right\}, \tag{2.1}$$

and unipotent radical

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \;\middle|\; b \in \mathrm{Sym}_n \right\}. \tag{2.2}$$

The stabilizer in $G(\mathbb{R}) = \mathrm{Sp}_n(\mathbb{R})$ of the point $i\cdot 1_n \in \mathfrak{H}_n$, the Siegel space of genus n, is the maximal compact subgroup

$$K = \left\{ k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; \mathbf{k} = A + iB \in U(n) \right\}. \tag{2.3}$$

Note that if $g = nm(a)k$, then $g = nm(a\mathbf{k}_0)k_0^{-1}k$, where $\mathbf{k}_0 \in O(n)$. In particular, in such a decomposition, we can always assume that $\det(a) > 0$. Let

$$\underline{w} = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}, \tag{2.4}$$

so that $\underline{w}$ corresponds to $i\,1_n \in U(n)$ and lies in the center of K. For $\tau \in \mathfrak{H}_n$ and $g \in G$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we let $j(g,\tau) = \det(c\tau + d)$ be the standard scalar automorphy factor. Note that $j(gk, i) = j(g,i)\,\det(\mathbf{k})^{-1}$.

2.2. **Weil representations.** Let V, $(\ ,\ )$ be a non-degenerate inner product space over $\mathbb{Q}$ of signature (p,q). If $\dim V = m = p+q$ is even, $\mathrm{Sp}_n(\mathbb{R}) \times \mathrm{O}(V(\mathbb{R}))$ acts on the space of Schwartz functions $\mathcal{S}(V(\mathbb{R})^n)$ via the Weil representation:

$$\begin{aligned} \omega(m(a))\varphi(x) &= \chi_V(\det(a))\,|\det(a)|^{\frac{m}{2}}\,\varphi(xa) \\ \omega(n(b))\varphi(x) &= e(\mathrm{tr}(Q(x)b))\,\varphi(x) \\ \omega(\underline{w})\varphi(x) &= \gamma(V)\int_{V(\mathbb{R})^n} e(\mathrm{tr}((x,y))\,\varphi(y)\,dy, \end{aligned}$$

and $\omega(h)\varphi(x) = \varphi(h^{-1}x)$ for $h \in O(V)(\mathbb{R})$. Here $\chi_V(t) = (\mathrm{sgn}(t))^{\frac{1}{2}(p-q)}$ and $\gamma(V) = e(\frac{1}{8}(p-q))$.

Let $D(V)$ be the space of oriented negative q-planes in $V(\mathbb{R})$. For $z \in D$, let $(\ ,\ )_z$ be the majorant of $(\ ,\ )$ defined by

$$(x,x)_z = (x,x) - 2(\mathrm{pr}_z(x), \mathrm{pr}_z(x)),$$

and let $\varphi_0(\cdot, z) \in \mathcal{S}(V(\mathbb{R})^n)$, given by

$$\varphi_0(x,z) = \exp(-\pi\,\mathrm{tr}((x,x)_z)),$$

be the associated Gaussian. It is an eigenfunction for K with

$$\omega(k)\varphi_0(\cdot,z) = \det(\mathbf{k})^{\frac{p-q}{2}}\,\varphi_0(\cdot,z).$$

2.3. **The degenerate principal series.** For $G = \mathrm{Sp}_n(\mathbb{R})$ and the Siegel parabolic $P = NM$, with notation as in Section 2.1, let $I^{\mathrm{sm}}(s,\chi)$ be the degenerate principal series representation given by right multiplication on the space of smooth functions ϕ on G with

$$\phi(n(b)m(a)g) = \chi(\det(a))\,|\det(a)|^{s+\rho}\,\phi(g), \tag{2.5}$$

where $\rho = \rho_n = \frac{1}{2}(n+1)$. In the case of interest to us, $\chi(t) = \mathrm{sgn}(t)^\nu$ for $\nu = 0$, 1. We let $I(s) = I(s,\chi)$ be the space of K-finite functions; it is the $(\mathfrak{g},K)$-module associated to $I^{\mathrm{sm}}(s)$. The structure of $I(s)$ is known, [16], [17], [18]. We review the facts that we need and refer the reader to these papers for more information.

2.4. **The infinitesimal character.** Let $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{k} = \mathrm{Lie}(K)_{\mathbb{C}}$ and hence also of $\mathfrak{g}$. Let $\mathfrak{z}(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ and let

$$\gamma : \mathfrak{z}(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^W$$

be the Harish-Chandra isomorphism, [10]. For $\lambda \in \mathfrak{h}^*$, let χ_λ be the character of $\mathfrak{z}(\mathfrak{g})$ given by $\chi_\lambda(Z) = \gamma(Z)(\lambda)$. Following the notation of [16], for $x = (x_1,\dots,x_n) \in \mathbb{C}^n$, we write

$$d(x) = \mathrm{diag}(x_1,\dots,x_n), \qquad h(x) = \begin{pmatrix} & -i\,d(x) \\ i\,d(x) & \end{pmatrix},$$

and take $\mathfrak{h} = \{h(x) \mid x \in \mathbb{C}^n\}$. Then $H_j = h(e_j)$ is a basis for $\mathfrak{h}$ with dual basis $\epsilon_j \in \mathfrak{h}^*$, $1 \le j \le n$. Then the infinitesimal character of $I(s)$ is $\chi_{\lambda(s)+\rho_G}$, where

$$\lambda(s) = (s-\rho)\sum_j \epsilon_j \tag{2.6}$$

and $\rho_G = \sum_j (n-j+1)\epsilon_j$, cf. [10], Theorem 4, p. 76, for example. Let C be the Casimir operator of $\mathfrak{g}$. Then C acts in $I(s)$ by the scalar

$$\chi_{\lambda(s)+\rho_G}(C) = \langle\, \lambda(s)+\rho_G, \lambda(s)+\rho_G \,\rangle - \langle\, \rho_G, \rho_G \,\rangle = \frac{1}{8}\,(s+\rho)(s-\rho). \tag{2.7}$$

This is consistent with the fact that the trivial representation of G is a constituent on $I(s)$ at the points $s = \pm\rho$.

Note that the Killing form on $\mathfrak{g} \subset M_{2n}(\mathbb{C})$ is given by

$$\langle\, X, Y \,\rangle_{\mathfrak{g}} = 4n\,\mathrm{tr}(XY),$$

so that, since $\mathrm{tr}(p_+(x)p_-(y)) = \mathrm{tr}(xy)$,

$$C = C_{\mathfrak{k}} + \frac{1}{4n}\sum_\alpha p_+(e_\alpha)\,p_-(e_\alpha^\vee) + p_-(e_\alpha^\vee)\,p_+(e_\alpha), \tag{2.8}$$

where $C_{\mathfrak{k}}$ is the $\mathfrak{k}$ component of C.

2.5. **K-types.** For further details, cf. [16]. As a representation of K, we have

$$I(s) \simeq \mathrm{Ind}_{M\cap K}^{K}(\chi) \simeq \mathrm{Ind}_{O(n)}^{U(n)}\mathrm{sgn}(\det)^\nu.$$

Thus the K-types of $I(s)$ have multiplicity one and an irreducible representation $(\sigma, \mathcal{V}_\sigma)$ of K occurs precisely when its highest weight has the form

$$(\ell_1, \dots, \ell_n), \quad \ell_1 \ge \dots \ge \ell_n, \qquad \ell_j \in \nu + 2\mathbb{Z},$$

or, equivalently, precisely when its restriction to $O(n) \simeq M \cap K$ contains the representation $(\det)^\nu$. For such σ,

$$\dim \mathrm{Hom}_K(\sigma, I(s)) = \dim[\, I(s) \otimes \sigma^\vee]^K = 1. \tag{2.9}$$

Suppose that $v_0 \in \sigma^\vee$ is an eigenvector for $O(n)$, so that $\sigma^\vee(k)v_0 = \det(k)^\nu v_0$ for all $k \in O(n)$. The vector v_0 is unique up to a non-zero scalar. A standard basis element for $[I(s) \otimes \sigma^\vee]^K$ is then given by

$$\phi_{s,\sigma}(nm(a)k) = \chi(\det(a))\,|\det(a)|^{s+\rho}\,\sigma^\vee(\mathbf{k}^{-1})\,v_0.$$

For example, for an integer ℓ, with $\ell \equiv \nu \mod 2$, the unique function $\phi_{s,\ell} \in I(s)$ with scalar K-type $\det(k)^\ell$ is given by

$$\phi_{s,\ell}(n(b)m(a)k) = \chi(\det(a))|\det(a)|^{s+\rho}\,\det(\mathbf{k})^\ell. \tag{2.10}$$

2.6. **Submodules.** For $s \notin \nu+2\mathbb{Z}$, the $(\mathfrak{g}, K)$-module $I(s)$ is irreducible. At points $s \in \nu+2\mathbb{Z}$, nontrivial submodules arise via the coinvariants for the Weil representation. For a quadratic space V over $\mathbb{R}$ of signature (p,q), $p+q$ even, with associated Weil representation $(\omega, S(V^n))$, for the additive character $x \mapsto e(x)$, there is an equivariant map

$$\lambda_V : S(V^n) \longrightarrow I(s_0), \qquad \varphi \mapsto (\omega(g)\varphi)(0),$$

where $s_0 = \frac{1}{2}(p+q) - \rho$ and $\nu \equiv \frac{1}{2}(p-q) \mod 2$. The image, $R(p,q)$, is the $(\mathfrak{g}, K)$-submodule of $I(s_0)$ generated by the scalar K-type $(\det)^\ell$ with $\ell = \frac{1}{2}(p-q)$. Moreover, the vector $\phi_{s_0, \frac{1}{2}(p-q)}$ is the image of the Gaussian $\varphi_V^0 \in S(V^n)$, where

$$\omega_V(k)\, \varphi_V^0 = \det(\mathbf{k})^{\frac{1}{2}(p-q)}\, \varphi_V^0.$$

For example, for signature $(m+2,0)$ with $m+2 > 2n+2$, $R(m+2,0) \subset I(s_0)$ is a holomorphic discrete series representation with scalar K-type $(\det)^\kappa$, $\kappa = \frac{m}{2}+1$. In particular, the vector $\phi_{s_0,\kappa}$ for $s_0 = \kappa - \rho$ is killed by $\mathfrak{p}_-$.

On the other hand, for signature $(m,2)$, the vector

$$\varphi_{KM} \in [S(V^n) \otimes \wedge^{(n,n)}(\mathfrak{p}_H^*)]^{K_H},$$

satisfies

$$\omega(k)\varphi_{KM} = \det(\mathbf{k})^\kappa\, \varphi_{KM}.$$

The image of this vector under the map $\lambda_{m,2}$ is again $\phi_{s_0,\kappa}$, so that we have submodules

$$R(m+2,0) \subset R(m,2) \subset I(s_0), \qquad s_0 = \kappa - \rho.$$

Note that, by [16], $R(p,q)$ is the largest quotient of $S(V^n)$ on which the orthogonal group $O(V) = O(p,q)$ acts trivially, i.e., the space of $O(V)$-coinvariants.

3. Goodman-Wallach operators

In our discussion of both conical and Whittaker vectors, we will only consider the degenerate principal series representation $I(s)$ and the Siegel parabolic $P = NM$. In this case, the simple example for $\mathrm{SL}_2(\mathbb{R})$ worked out in the introduction of Goodman-Wallach [13] provides an adequate template. An essential feature is that the various classical special functions occurring there are replaced by their matrix argument generalizations. These results seem to be new; at least we could not find such explicit formulas in the literature. Since we work with intertwining operators which express our functions via integral representations, we derive, as a consequence, the behavior of our functions under the differential operators coming from the center of the enveloping algebra $\mathfrak{z}(\mathfrak{g})$, whereas, in the case of $\mathrm{SL}_2(\mathbb{R})$ one can work with classical solutions of the second order ode satisfied by the radial part. For discussion of the more general theory including general results about the existence of Goodman-Wallach operators, cf. [20].

3.1. **Conical and Whittaker vectors.**

Suppose that $(\pi, \mathcal{V})$ is a continuous representation of G on a Banach space $\mathcal{V}$ and that

$$F_{\mathrm{con}} : \mathcal{V} \longrightarrow I^{\mathrm{an}}(s)$$

is a G-equivariant linear map. Here $I^{\mathrm{an}}(s)$ is the space of real analytic functions on G satisfying (2.5). The linear functional $\mu_{\mathrm{con}} \in \mathcal{V}^*$ defined by $\mu_{\mathrm{con}}(v) = F_{\mathrm{con}}(v)(e)$ satisfies

$$\mu_{\mathrm{con}}(\pi(nm(a))v) = \chi(\det(a))\, |\det(a)|^{s+\rho}\, \mu_{\mathrm{con}}(v).$$

We refer to such a vector as a conical vector[2] in $\mathcal{V}^*$ of type $(P, s+\rho)$. Conical vectors in the dual $I^{\mathrm{an}}(s)^*$ are given by

$$c_1(\phi) = \phi(e), \tag{3.1}$$

and

$$c_{\underline{w}}(\phi) = (A(s,\underline{w})\phi)(e) = \int_N \phi(\underline{w}n(b))\,db, \tag{3.2}$$

where, for $\mathrm{Re}(s) > \rho$, the intertwining operator $A(s,\underline{w}) : I(s) \longrightarrow I(-s)$ is defined by the integral

$$(A(s,\underline{w})\phi)(g) = \int_N \phi(\underline{w}ng)\,dn, \tag{3.3}$$

for $\underline{w}$ given by (2.4). It has a meromorphic analytic continuation. According to our terminology, these are of type $(P, s+\rho)$ and $(P, -s+\rho)$ respectively.

Similarly, for $T \in \mathrm{Sym}_n(\mathbb{C})$, suppose that

$$F_{\mathrm{wh}} : \mathcal{V} \longrightarrow \mathcal{W}^T(G)^{\mathrm{an}}$$

is a G-equivariant linear map, where $\mathcal{W}^T(G)^{\mathrm{an}}$ is the space of real analytic functions on G satisfying (1.4). The linear functional $\mu_{\mathrm{wh}} \in \mathcal{V}^*$ defined by $\mu_{\mathrm{wh}}(v) = F_{\mathrm{wh}}(v)(e)$ satisfies

$$\mu_{\mathrm{wh}}(\pi(n(b))v) = e(\mathrm{tr}(Tb))\,\mu_{\mathrm{wh}}(v).$$

We refer to such a vector in $\mathcal{V}^*$ as a Whittaker vector (of type (N,T)).

We can make analogous definitions for $\mathcal{V}$ an irreducible $U(\mathfrak{g})$-module, and now explain an essential idea of [13] relating conical and Whittaker vectors in $\mathcal{V}^*$. Our goal is to motivate the explicit construction given below; for a more careful treatment cf. [13] and [20].

Let $\mathfrak{n} = \mathrm{Lie}(N)_{\mathbb{C}}$, $\mathfrak{m} = \mathrm{Lie}(M)_{\mathbb{C}}$, and let $\bar{\mathfrak{n}} = \mathrm{Lie}(\bar{N})_{\mathbb{C}}$ where $\bar{N}$ is the unipotent radical of the opposite maximal parabolic $\bar{P} = M\bar{N}$. Let $\mathbb{C}(s+\rho)$ be the one dimensional representation of $\mathfrak{m}$ determined by the representation $|\det|^{s+\rho}$ of M and extend it to a representation of $\mathfrak{m}+\mathfrak{n}$, trivial on $\mathfrak{n}$. Define the generalized Verma modules

$$V(P, s+\rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{m}+\mathfrak{n})} \mathbb{C}(s+\rho)$$

and

$$\bar{V}(\bar{P}, -s-\rho) = \mathbb{C}(-s-\rho) \otimes_{U(\mathfrak{m}+\bar{\mathfrak{n}})} U(\mathfrak{g}).$$

By the Poincaré-Birkoff-Witt theorem,

$$V(P, s+\rho) = U(\bar{\mathfrak{n}}) \otimes_{\mathbb{C}} \mathbb{C}(s+\rho), \quad \text{and} \quad \bar{V}(\bar{P}, -s-\rho) = \mathbb{C}(-s-\rho) \otimes_{\mathbb{C}} U(\mathfrak{n}).$$

There is a natural pairing

$$\langle\langle\ ,\ \rangle\rangle : \bar{V}(\bar{P}, -s-\rho) \otimes_{\mathbb{C}} V(P, s+\rho) \longrightarrow \bar{V}(\bar{P}, -s-\rho) \otimes_{U(\mathfrak{g})} V(P, s+\rho) \xrightarrow{\sim} \mathbb{C},$$

and, for $Z \in U(\mathfrak{g})$, $\bar{u} \in \bar{V}(\bar{P}, -s-\rho)$ and $u \in V(P, s+\rho)$,

$$\langle\langle u^* Z, u\rangle\rangle = \langle\langle u^*, Zu\rangle\rangle.$$

[2]Here we are introducing a compressed version of the standard terminology, [13] and [20], convenient for our special case.

This pairing is non-degenerate precisely when $V(P, s+\rho)$ is irreducible, [20], Section 3.1. For the rest of our discussion, we suppose that this is the case. Note that, since $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are abelian

$$U(\bar{\mathfrak{n}}) = S(\bar{\mathfrak{n}}) = \bigoplus_{d\geq 0} S(\bar{\mathfrak{n}})_d \tag{3.4}$$

is the symmetric algebra on $\bar{\mathfrak{n}}$, graded by degree, and the completion

$$S(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]} = U(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]} = \varprojlim U(\bar{\mathfrak{n}})/\bar{\mathfrak{n}}^k U(\bar{\mathfrak{n}}) \simeq \prod_{d\geq 0} S(\bar{\mathfrak{n}})_d, \tag{3.5}$$

is the ring of formal power series in such elements. Let

$$\hat{V}(P, s+\rho) = V(P, s+\rho)_{[\bar{\mathfrak{n}}]} = U(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]} \otimes_{\mathbb{C}} \mathbb{C}(s+\rho)$$

be the $\bar{\mathfrak{n}}$-completion of $V(P, s+\rho)$. The pairing $\langle\langle\, , \rangle\rangle$ extends to this space and induces an isomorphism

$$\hat{V}(P, s+\rho) \xrightarrow{\sim} \bar{V}(\bar{P}, -s-\rho)^*. \tag{3.6}$$

Then, following the notation of [20], for an admissible[3] character ψ of $\mathfrak{n}$, the spaces

$$\mathrm{Wh}^*_{\mathfrak{n},\psi}(\bar{V}(\bar{P}, -s-\rho)) = \{\, w \in \bar{V}(\bar{P}, -s-\rho)^* \mid X\cdot w = \psi(X)\, w,\ \forall X \in \mathfrak{n}\}, \tag{3.7}$$

and

$$\mathrm{Wh}_{\mathfrak{n},\psi}(\hat{V}(P, s+\rho)) = \{\, w \in \hat{V}(P, s+\rho) \mid X\cdot w = \psi(X)\, w,\ \forall X \in \mathfrak{n}\} \tag{3.8}$$

are isomorphic via (3.6). If we extend ψ to an algebra homomorphism $\psi : U(\mathfrak{n}) \longrightarrow \mathbb{C}$, then there is an obvious basis for the space (3.7) given by the functional $1\otimes\psi$ on $\bar{V}(\bar{P}, -s-\rho) \simeq \mathbb{C}(-s-\rho)\otimes_{\mathbb{C}} U(\mathfrak{n})$. We write gw^ψ_s for the corresponding element of $\hat{V}(P, s+\rho)$, viewed as a formal power series in the elements of $\bar{\mathfrak{n}}$, and we refer to it as the Goodman-Wallach element. It is characterized by

$$\langle\langle A, \mathrm{gw}^\psi_s\rangle\rangle = \psi(A), \qquad \text{for all } A \in U(\mathfrak{n}).$$

Returning to the irreducible $U(\mathfrak{g})$-module $(\pi, \mathcal{V})$, we let $U(\mathfrak{g})$ act on $\mathcal{V}^*$ on the left by $Z\cdot\mu = \mu\cdot {}^tZ$, where ${}^t : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ is the involution restricting to $X \mapsto -X$ on $\mathfrak{g}$. We assume that $\mathcal{V}$ is finitely generated as a $U(\mathfrak{n})$-module. For a conical vector $\mu \in \mathcal{V}^*$ of type $(P, s+\rho)$, we have a homomorphism

$$\hat{V}(P, -s-\rho) \longrightarrow \mathcal{V}^*, \qquad Z \mapsto Z\cdot\mu.$$

Then, for the Goodman-Wallach element $\mathrm{gw}^\psi_s \in \hat{V}(P, -s-\rho)$, and for $X \in \mathfrak{n}$, we have

$$(\mathrm{gw}^\psi_s\cdot\mu)\cdot X = (\mu\cdot {}^t\mathrm{gw}^\psi_s)\cdot X = \mu\cdot {}^t(-X\mathrm{gw}^\psi_s) = -\psi(X)\,\mathrm{gw}^\psi_s\cdot\mu.$$

Thus, $\mathrm{gw}^\psi_s\cdot\mu$ is a Whittaker vector of type $(N, -\psi)$. Of course, we have only given a rough sketch of the idea here, including the restriction to the case where $V(P, s+\rho)$ is irreducible. After removing this restriction, a main point of the Goodman-Wallach theory is to give estimates on the growth of the components of the power series gw^ψ_s so that it can be used to define G-intertwining operators – the Goodman-Wallach operators – from principal series representations to spaces of Whittaker functions preserving Gevrey classes, i.e., certain

[3] In our situation, for $X \in \mathfrak{n} \simeq \mathrm{Sym}_n(\mathbb{C})$, and $\psi(X) = \mathrm{tr}(TX)$, this simply means that $\det(T) \neq 0$.

function spaces between real analytic and C^∞, cf. the remark on p. 228 of [20] for a more precise statement.

3.2. **An explicit formula for the Goodman-Wallach operator for** $\mathrm{Sp}_n(\mathbb{R})$**.** We now describe the matrix argument Bessel function giving the Goodman-Wallach operator.

For $\phi \in I(s)$, the function $\Psi(\phi)$ defined by (1.12) and its Fourier transform (1.13) satisfy

$$\Psi(x; \pi(m(a))\phi) = \det(a)^{s+\rho}\, \Psi({}^t a x a; \phi), \tag{3.9}$$

and

$$\hat{\Psi}(y; \pi(m(a))\phi) = \det(a)^{s-\rho}\, \hat{\Psi}(a^{-1} y\, {}^t a^{-1}; \phi). \tag{3.10}$$

The conical vectors c_1 and $c_{\underline{w}}$ defined by (3.1) and (3.2) can be written as

$$\langle\, c_1, \phi\,\rangle = \int_S \hat{\Psi}(y; \phi)\, dy = \Psi(0; \phi), \tag{3.11}$$

and

$$\langle\, c_{\underline{w}}, \phi\,\rangle = \int_S \Psi(y; \pi(\underline{w})\phi)\, dy = \hat{\Psi}(0; \pi(\underline{w})\phi). \tag{3.12}$$

Here note that

$$\int_S \phi(\underline{w} n(b))\, db = \int_S \phi(n_-(-b)\underline{w})\, db = \int_S \phi(n_-(b)\underline{w})\, db.$$

We next define the relevant Bessel type function on S, specializing some of the notation of [12] and [22] to the present case. This notation is summarized in Appendix 1, which the reader should consult for things not explained here.

For z and $w \in S_{\mathbb{C}}$, the function $\mathrm{GW}_s(z, w)$ defined by (1.15) coincides with the J-Bessel function

$$\mathrm{GW}_s(z, w) = \mathcal{J}_{s+\rho}(z, w), \tag{3.13}$$

in the notation of [22], p. 818 and p. 823. Note that $\Phi_{\mathbf{m}}(z, w)$ is the function on $S_{\mathbb{C}} \times S_{\mathbb{C}}$ described in [22], Lemma 1.11. In particular, this function is holomorphic in z, antiholomorphic in w, and satisfies $\Phi_{\mathbf{m}}(z, 1_n) = \Phi_{\mathbf{m}}(z)$, where $\Phi_{\mathbf{m}}(z)$ is the spherical polynomial in [12], Chapter XI, Section 3. Also recall that $\Phi_{\mathbf{m}}(z)$ is homogeneous of degree $|\mathbf{m}| = \sum_i m_i$. Moreover, for $a \in \mathrm{GL}_n(\mathbb{C})$,

$$\Phi_{\mathbf{m}}(a \cdot z, w) = \Phi_{\mathbf{m}}(z, {}^t\bar{a} \cdot w), \tag{3.14}$$

and $\overline{\Phi_{\mathbf{m}}(z, w)} = \Phi_{\mathbf{m}}(w, z)$. The invariance (3.14) is inherited by $\mathrm{GW}_s(z, w)$.

Theorem 3.1. *For $\phi \in I(s)$, let*

$$\omega_1(\phi) = \int_S \mathrm{GW}_s(2\pi y, 2\pi w)\, \hat{\Psi}(y; \phi)\, dy. \tag{3.15}$$

For $X \in \mathrm{Sym}_n(\mathbb{R})$, let

$$n_+(X) = \begin{pmatrix} 0 & X \\ & 0 \end{pmatrix}.$$

Then,

$$\omega_1(\pi(n_+(X))\phi) = 2\pi i\,\mathrm{tr}(X\bar{w})\,\omega_1(\phi). \tag{3.16}$$

Corollary 3.2. *For $Y \in \mathrm{Sym}_n(\mathbb{C}) \simeq \bar{\mathfrak{n}}$, view $\mathrm{GW}_s(2\pi Y, 2\pi w)$ as a power series[4] in $S(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]}$. Then this power series defines the Goodman-Wallach-Matumoto operator in $\hat{V}(P, -s-\rho)$ for the Siegel parabolic P of $G = \mathrm{Sp}_n(\mathbb{R})$ and the character $n_+(X) \mapsto 2\pi i\,\mathrm{tr}(X\bar{w})$ of $\mathfrak{n}$.*

Definition 3.3. *For $T \in \mathrm{Sym}_n(\mathbb{R})$ with $\det(T) \neq 0$, let ω_1^T be the Whittaker functional constructed from the conical vector c_1, so that*

$$\omega_1^T(\phi) = \int_S \mathrm{GW}_s(2\pi y, 2\pi T)\,\hat{\Psi}(y;\phi)\,dy. \tag{3.17}$$

and

$$\omega_1^T(\phi)(n(b)g) = e(\mathrm{tr}(Tb))\,\omega_1^T(\phi)(g).$$

3.3. **Proof of the Goodman-Wallach identity.** We want to prove (3.16) and so we consider

$$\omega_1(n_+(X)\phi) = \int_S \mathrm{GW}_s(2\pi y, 2\pi w)\,\hat{\Psi}(y; n_+(X)\phi)\,dy. \tag{3.18}$$

We adopt some of the notation and setup from [12] and [22]. Recall that S is a simple Euclidean Jordan algebra with product $x \cdot y = \frac{1}{2}(xy + yx)$. Endomorphisms $P(x)$ and $P(x,y)$ of S are defined by, [22], p.794,

$$P(x)z = xzx, \qquad P(x,y)z = xzy + yzx.$$

Let e_α be a basis for S and let $e_\alpha^\vee$ be the dual basis with respect to the trace form, cf. Appendix 1. In particular, we write $x = \sum_\alpha x_\alpha e_\alpha$. Define vector-valued differential operators as follows. The gradient operator

$$\frac{\partial}{\partial x} = \sum_\alpha \frac{\partial}{\partial x_\alpha}\, e_\alpha^\vee,$$

is characterized by

$$\partial_a = \mathrm{tr}(a\frac{\partial}{\partial x}),$$

where, for $a \in S$, ∂_a is the directional derivative associated to the constant vector field a. For a complex scalar λ, the Bessel operator $\mathcal{B}_\lambda$ is defined by

$$\mathcal{B}_\lambda = P(\frac{\partial}{\partial x})x + \lambda\,\frac{\partial}{\partial x} = \frac{\partial}{\partial x}x\frac{\partial}{\partial x} + \lambda\,\frac{\partial}{\partial x}.$$

Thus, if f is a C^2 function on S, then $\mathcal{B}_\lambda f$ is an S-valued function on S.

The key fact is the following (see also Proposition 3.3 in [15]):

Proposition 3.4. *For X and $Y \in \mathrm{Sym}_n(\mathbb{R})$,*

(i)

$$\Psi(x; \pi(n_+(X))\phi) = -\mathrm{tr}(X\big(s\,x + x\frac{\partial}{\partial x}x\big)\Psi(x;\phi)),$$

[4]Concretely, write $Y = \sum_\alpha Y_\alpha\, e_\alpha^\vee$ and view $\mathrm{GW}_s(2\pi Y, w)$ as a power series in the Y_α's.

(ii)

$$\Psi(x;\pi(n_-(Y))\phi) = \mathrm{tr}(Y\frac{\partial}{\partial x})\Psi(x;\phi),$$

(iii)

$$\hat{\Psi}(y;\pi(n_+(X))\phi) = \frac{1}{2\pi i}\,\mathrm{tr}(X\mathcal{B}_{-s}\hat{\Psi}(y;\phi)),$$

(iv)

$$\hat{\Psi}(y;\pi(n_-(Y))\phi) = -2\pi i\,\mathrm{tr}(Yy)\,\hat{\Psi}(y;\phi).$$

Proof. First we note that

$$\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}1 & X\\ & 1\end{pmatrix} = \begin{pmatrix}1 & X\\ x & 1+xX\end{pmatrix} = \begin{pmatrix}1 & *\\ & 1\end{pmatrix}\begin{pmatrix}{}^tA^{-1} & \\ & A\end{pmatrix}\begin{pmatrix}1 & \\ A^{-1}x & 1\end{pmatrix},$$

where $A = 1+xX$ and $* = X(1+xX)^{-1}$. Here we are going to take tX in place of X so that $1+xX$ will be invertible for t sufficiently small. Thus, in this range,

$$\phi(n_-(x)n_+(X)) = \det(1+xX)^{-s-\rho}\,\phi(n_-((1+xX)^{-1}x)).$$

We now replace X by tX and take $d/dt|_{t=0}$. First we have

$$\frac{d}{dt}\det(1+txX)^{-s-\rho}|_{t=0} = -(s+\rho)\,\mathrm{tr}(xX),$$

where we note that

$$\det(1+txX) = 1+\mathrm{tr}(txX)+O(t^2).$$

Next, we let $z = (1+xtX)^{-1}x$ and compute

$$0 = \frac{d}{dt}\big((1+xtX)z\big)|_{t=0} = xXx + \frac{dz}{dt}|_{t=0},$$

so that, writing $z = \sum_\alpha z_\alpha e_\alpha$, we have

$$\frac{dz}{dt}|_{t=0} = \sum_\alpha \frac{dz_\alpha}{dt}e_\alpha = -\sum_\alpha (xXx)_\alpha e_\alpha.$$

Therefore

$$\frac{d}{dt}\phi(z)|_{t=0} = \sum_\alpha \frac{\partial\phi}{\partial z_\alpha}\,\frac{dz_\alpha}{dt}|_{t=0} = -\mathrm{tr}(xXx\frac{\partial}{\partial x})\phi.$$

Thus we have proved that

$$\Psi(x;\pi(n_+(X))\phi) = -\Big(\ (s+\rho)\,\mathrm{tr}(Xx) + \mathrm{tr}(xXx\frac{\partial}{\partial x})\ \Big)\Psi(x;\phi).$$

But we have[5]

$$\mathrm{tr}(xXx\frac{\partial}{\partial x}) = \mathrm{tr}(Xx\frac{\partial}{\partial x}x) - \rho\,\mathrm{tr}(xX).$$

This gives (i).

[5]Indeed, writing $\bullet$ for the 'evaluation' product, we have

$$(\sum_\alpha \frac{\partial}{\partial x_\alpha}\,e_\alpha^\vee)\bullet(\sum_\alpha x_\alpha e_\alpha) = \sum_\alpha e_\alpha^\vee\,e_\alpha = \frac{1}{2}(n+1)\sum_i e_{ii}.$$

Now we consider the Fourier transform

$$-\int_S e(\mathrm{tr}(xy))\left(s\,x + x\frac{\partial}{\partial x}x\right)\Psi(x;\phi)\,dx.$$

Note that if we write $y = \sum_\alpha y_\alpha e_\alpha^\vee$, then $(\partial/\partial y) = \sum_\alpha (\partial/\partial y_\alpha)\,e_\alpha$, and we have

$$\frac{\partial}{\partial y}e(\mathrm{tr}(xy)) = (\sum_\alpha \frac{\partial}{\partial y_\alpha}\,e_\alpha)(e(\sum_\alpha x_\alpha y_\alpha)) = 2\pi i\sum_\alpha x_\alpha e_\alpha = 2\pi i x.$$

Then the factor $s\,x$ (resp. x^2) can be obtained by applying

$$\frac{s}{2\pi i}\frac{\partial}{\partial y}, \qquad (\text{resp. } (2\pi i)^{-2}\left(\frac{\partial}{\partial y}\right)^2)$$

outside the integral. Also

$$\int_S e(\mathrm{tr}(xy))\frac{\partial}{\partial x}\Psi(x;\phi)\,dx = -2\pi i\,y\int_S e(\mathrm{tr}(xy))\,\Psi(x;\phi)\,dx.$$

We obtain

$$\begin{aligned}
&-\int_S e(\mathrm{tr}(xy))\left(s\,x + x\frac{\partial}{\partial x}x\right)\Psi(x;\phi)\,dx \\
&= -\frac{1}{2\pi i}\left(s\,\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\,y\,\frac{\partial}{\partial y}\right)\int_S e(\mathrm{tr}(xy))\,\Psi(x;\phi)\,dx \\
&= \frac{1}{2\pi i}\,\mathcal{B}_{-s}\,\widehat{\Psi}(y;\phi).
\end{aligned}$$

This proves (iii) of Proposition 3.4. The proofs of (ii) and (iv) are easy and omitted. □

Now we return to (3.18) and, using (ii) of the previous proposition, obtain

$$\begin{aligned}
\omega_1(\pi(n_+(X))\phi) &= \int_S \mathrm{GW}_s(2\pi y, 2\pi w)\,\hat{\Psi}(y;\pi(n_+(X))\phi)\,dy \\
&= \frac{1}{2\pi i}\mathrm{tr}(X\int_S \mathrm{GW}_s(2\pi y, 2\pi w)\,\mathcal{B}_{-s}\,\hat{\Psi}(y;\phi)\,dy\ ) \\
&= \frac{1}{2\pi i}\mathrm{tr}(X\int_S \mathcal{B}_{-s}^*\,\mathrm{GW}_s(2\pi y, 2\pi w)\,\hat{\Psi}(y;\phi)\,dy\ ) \\
&= \frac{1}{2\pi i}\mathrm{tr}(X\int_S \mathcal{B}_s\,\mathrm{GW}_s(2\pi y, 2\pi w)\,\hat{\Psi}(y;\phi)\,dy\ ).
\end{aligned}$$

Here we use the fact that the adjoint of $\mathcal{B}_{-s}$ is $\mathcal{B}_s$. But now by (3.13) and Proposition 3.6 of [22], we have

$$\mathcal{B}_s\,\mathrm{GW}_s(2\pi y, 2\pi w) = -(2\pi)^2\,\bar{w}\cdot\mathrm{GW}_s(2\pi y, 2\pi w).$$

so that we obtain

$$\omega_1(n_+(X)\phi) = 2\pi i\,\mathrm{tr}(X\bar{w})\,\omega_1(\phi),$$

as required. This proves Theorem 3.1.

3.4. **Proof of Corollary 3.2.** To show that $\mathrm{GW}_s(2\pi Y, 2\pi w)$ is indeed the Goodman-Wallach element, as claimed, we proceed as follows. The pairing

$$\langle\langle\,,\rangle\rangle : \bar{V}(\bar{P}, s+\rho)\otimes_{\mathbb{C}} \hat{V}(P, -s-\rho) \longrightarrow \mathbb{C},$$

is characterized by

$$\langle\langle A\cdot Z, B\rangle\rangle = \langle\langle A, Z\cdot B\rangle\rangle,$$

for $A\in \bar{V}(\bar{P}, s+\rho)$, $B\in \hat{V}(P,-s-\rho)$, and $Z\in U(\mathfrak{g})$, and

$$\langle\langle 1\otimes n_+(X), n_-(Y)\otimes 1\rangle\rangle = (s+\rho)\,\mathrm{tr}(XY).$$

For any function $\phi\in I(s)$, and for $A\in S(\mathfrak{n})$ and $B\in U(\mathfrak{g})$, we define

$$\langle\langle A, B\rangle\rangle_\phi = -\int_S \hat{\Psi}(y; {}^t(A\cdot B)\cdot\phi)\,dy = -\Psi(0; {}^t(A\cdot B)\cdot\phi).$$

As a function of B this map factors through $V(P,-s-\rho)$ and $\langle\langle A,B\rangle\rangle_\phi = \langle\langle 1, A\cdot B\rangle\rangle_\phi$. Recall here that, as in 3.1, $A\mapsto {}^tA$ is the involution of $U(\mathfrak{g})$ which is -1 on $\mathfrak{g}$. Moreover, we have

$$\begin{aligned}
\langle\langle n_+(X), n_-(Y)\rangle\rangle_\phi &= -\Psi(0; n_-(-Y)\,n_+(-X)\cdot\phi)\\
&= -\Psi(0; ([n_-(-Y), n_+(-X)] + n_+(-X)\,n_-(-Y))\cdot\phi)\\
&= (s+\rho)\,\mathrm{tr}(XY)\,\Psi(0;\phi).
\end{aligned}$$

Taking ϕ with $\Psi(0;\phi)=\phi(e)=1$, we have $\langle\langle\,,\rangle\rangle_\phi = \langle\langle\,,\rangle\rangle$ on $U(\mathfrak{n})\times V(P,-s-\rho)$. Now by (iv) of Proposition 3.4, we take a power series $\mathrm{gw}_s\in S(\bar{\mathfrak{n}})_{[\bar{\mathfrak{n}}]}$ so that, for all ϕ,

$$\hat{\Psi}(y; {}^t\mathrm{gw}_s\cdot\phi) = \mathrm{GW}_s(2\pi y, 2\pi w)\,\hat{\Psi}(y;\phi).$$

Then

$$\begin{aligned}
\langle\langle A, \mathrm{gw}_s\rangle\rangle &= -\int_S \hat{\Psi}(y; {}^t(A\cdot\mathrm{gw}_s)\cdot\phi)\,dy\\
&= -\int_S \mathrm{GW}_s(2\pi y, 2\pi w)\,\hat{\Psi}(y; {}^tA\phi)\,dy\\
&= -\omega_1({}^tA\phi)\\
&= -\psi_{-2\pi i\bar{w}}(A)\,\omega_1(\phi)\\
&= \psi_{-2\pi i\bar{w}}(A)\,\langle\langle 1, \mathrm{gw}_s\rangle\rangle,
\end{aligned}$$

where $\psi_{-2\pi i\bar{w}} : S(\mathfrak{n})\longrightarrow\mathbb{C}$ is the character determined by $n_+(X)\mapsto -2\pi i\,\mathrm{tr}(X\bar{w})$. This in fact shows that the Goodman-Wallach element is actually given by

$$\mathrm{gw}_s^\natural = \langle\langle 1, \mathrm{gw}_s\rangle\rangle^{-1}\,\mathrm{gw}_s.$$

4. Calculation of the 'bad' Whittaker function: the scalar case

In this section, we determine the Whittaker function $\omega_1^T(\phi_{s,\ell})$ for $\phi_{s,\ell}\in I(s)$ with a scalar K-type and for $\epsilon T\in\mathrm{Sym}_n(\mathbb{R})_{>0}$ and $\epsilon=\pm1$. Whittaker functions for other K-types can then be obtained by applying differential operators. In the case of $\mathrm{SL}_2(\mathbb{R})$, i.e., for $n=1$, the radial part of the Whittaker function we obtain is essentially the classical confluent hypergeometric function $M(a,b;z)$ in the notation of [1], for example. On the other hand, again for $n=1$, the Whittaker functional obtained by applying the Goodman-Wallach operator to the conical vector c_w yields a Whittaker function whose radial part is the classical function $U(a,b;z)$. Thus, this traditional basis for the solution space to the Whittaker ode arises in a natural way from the pair of conical vectors c_1 and c_w. The calculation of this section constructs the analogue of the M-Whittaker function for $\mathrm{Sp}_n(\mathbb{R})$. As noted earlier, we do not have a corresponding evaluation for T of arbitrary signature, and we will indicate in the course of the calculation where the assumption that T is definite is used.

For the standard vector $\phi_{s,\ell}$ with scalar K-type defined in (2.10), the Whittaker function

$$f^T_{s,\ell}(g) := \omega_1^T(\pi(g)\phi_{s,\ell}) = \int_S \mathrm{GW}_s(2\pi y, 2\pi\, T)\,\hat{\Psi}(y;\pi(g)\phi_{s,\ell})\,dy \tag{4.1}$$

satisfies

$$f^T_{s,\ell}(n(b)m(a)k) = e(\mathrm{tr}(Tb))\,f^T_{s,\ell}(m(a))\,\det(\mathbf{k})^\ell, \tag{4.2}$$

and hence is determined by its restriction to $\mathrm{GL}_n(\mathbb{R})$. Note that we will frequently omitted the T as a superscript to lighten the notation. By analogy with the one variable case, for $z \in S_{\mathbb{C}}$ and the a and $b \in \mathbb{C}$ with $\mathrm{Re}(a) > \rho - 1$, $\mathrm{Re}(b) > \rho - 1$, we let

$$M_n(a,b;z) = \frac{\Gamma_n(b)}{\Gamma_n(a)\Gamma_n(b-a)} \int\limits_{\substack{t>0\\1-t>0}} e^{\mathrm{tr}(zt)}\,\det(t)^{a-\rho}\,\det(1-t)^{b-a-\rho}\,dt. \tag{4.3}$$

This is the standard matrix argument hypergeometric function

$$M_n(a,b;z) = {}_1F_1(a;b;z). \tag{4.4}$$

as in [14], [12], etc. Our first main result is the following.

Theorem 4.1. *Let $v = a\,{}^ta$, $\alpha = \frac{1}{2}(s+\rho-\epsilon\ell)$ and $\beta = \frac{1}{2}(s+\rho+\epsilon\ell)$. Then, writing $\epsilon\, T = {}^tcc$,*

$$f_{s,\ell}(m(a)) = 2^{n(s-\rho+1)}\,\det(v)^{\frac{1}{2}(s+\rho)}\,\exp(-2\pi\mathrm{tr}(\epsilon\, Tv))\,M_n(\alpha,\alpha+\beta;4\pi cv\,{}^tc).$$

For future reference, we give the full formula

$$f_{s,\ell}(n(b)m(a)k) = c(n,s)\,\det(v)^{\frac{1}{2}(s+\rho)}\,e(\mathrm{tr}(Tb))\exp(-2\pi\mathrm{tr}(\epsilon\, Tv))\,M_n(\alpha,\alpha+\beta;4\pi cv\,{}^tc)\,\det(\mathbf{k})^\ell, \tag{4.5}$$

where

$$c(n,s) = 2^{n(s-\rho+1)}. \tag{4.6}$$

Remark 4.2. *The hypergeometric function $M_n(\alpha,\alpha+\beta;z)$ is given by a power series which is everywhere convergent in z provided the values of s for which some factor $(\alpha+\beta)_{\mathbf{m}} = (s+\rho)_{\mathbf{m}}$ in the denominator vanishes are excluded,* [12]. *The excluded values are thus $\mathbb{Z}_{<0}$ for $n=1$ and*

$$\mathbb{Z}_{<0} \cup \big(-\frac{1}{2} + \mathbb{Z}_{<0}\big)$$

for $n \geq 2$.

By construction, the Whittaker function $f_{s,\ell}$ is an eigenfunction for the center $\mathfrak{z}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ with the same eigencharacter as $I(s)$.

Corollary 4.3. *For all $Z \in \mathfrak{z}(\mathfrak{g})$,*

$$Z \cdot f_{s,\ell} = \chi_{\lambda(s)+\rho_G}(Z)\, f_{s,\ell},$$

where $\chi_{\lambda(s)+\rho_G}$ is the character of $\mathfrak{z}(\mathfrak{g})$ given by (2.6). In particular, for the Casimir operator C, by (2.7),

$$C \cdot f_{s,\ell} = \frac{1}{8}\,(s+\rho)(s-\rho)\, f_{s,\ell}.$$

Proof of Theorem 4.1. This amounts to a long calculation. We first simplify by eliminating T. Writing $2\pi\,\epsilon\,T = c^2$ for $c = {}^t c > 0$, we have, by (3.14),

$$\mathrm{GW}_s(2\pi y, 2\pi\,T) = \mathrm{GW}_s(2\pi \epsilon^t cyc, 1_n).$$

Then, using (3.10), and setting $\mathrm{GW}_s(z) = \mathrm{GW}_s(z, 1_n)$, we have

$$\int_S \mathrm{GW}_s(2\pi y, 2\pi T)\,\hat{\Psi}(y; \pi(m(a))\phi)\, dy = |\det(2\pi T)|^{-\frac{1}{2}(s+\rho)} \int_S \mathrm{GW}_s(2\pi\epsilon y)\,\hat{\Psi}(y; \pi(m(ca))\phi)\, dy.$$

Remark 4.4. *Note that it is at this point that we use the fact that T is definite in an essential way.*

Therefore, it suffices to compute

$$\omega_1^{\epsilon}(\pi(m(a))\phi) := \int_S \mathrm{GW}_s(2\pi\epsilon y)\,\hat{\Psi}(y; \pi(m(a))\phi)\, dy. \tag{4.7}$$

We write this as

$$\omega_1^{\epsilon}(\pi(m(a))\phi) = \sum_{p+q=n} \omega_1^{\epsilon}(\pi(m(a))\phi)_{p,q},$$

where

$$\omega_1^{\epsilon}(\pi(m(a))\phi)_{p,q} = \int_{S_{p,q}} \mathrm{GW}_s(2\pi\epsilon y)\,\hat{\Psi}(y; \pi(m(a))\phi)\, dy \tag{4.8}$$

for $S_{p,q}$ the subset of invertible matrices in S of signature (p,q).

Specializing to the case $\phi = \phi_{s,\ell}$, the first step is the following.

Proposition 4.5. *For $y \in S_{p,q}$,*

$$\begin{aligned}\hat{\Psi}(y; \pi(m(a))\phi_{\ell,s}) &= \frac{(2\pi)^{n\rho}\, 2^{-n(\rho-1)}}{\Gamma_n(\alpha)\Gamma_n(\beta)}\, \det(v)^{-\frac{1}{2}(s+\rho)}\, |\det(y)|^s\, (2\pi)^{ns}\, 2^{ns} \\ &\quad\times \int_{\substack{x+\epsilon_p>0\\ x+\epsilon'_q>0}} e^{-2\pi \mathrm{tr}({}^t c v^{-1} c\,(1+2x))} \det(x+\epsilon_p)^{\alpha-\rho} \det(x+\epsilon'_q)^{\beta-\rho}\, dx,\end{aligned} \tag{4.9}$$

where $y = c\epsilon_{p,q}{}^t c$ and $v = a^t a$. Here, as in [23]*, $\alpha = \frac{1}{2}(s+\rho+\ell)$, $\beta = \frac{1}{2}(s+\rho-\ell)$, $\epsilon_p = \mathrm{diag}(1_p, 0)$, and $\epsilon'_q = \mathrm{diag}(0, 1_q)$.*

Proof. If we write $n_-(x) = nmk$ for $k = k(n_-(x)) \in K$, then

$$\Psi(x;\phi) = \det(1+x^2)^{-\frac{1}{2}(s+\rho)}\phi(k(n_-(x))).$$

and

$$\Psi(x;\phi_{\ell,s}) = \det(1+ix)^{-\alpha}\,\det(1-ix)^{-\beta},$$

where we are now using Shimura's convention where $\alpha = \frac{1}{2}(s+\rho+\ell)$ and $\beta = \frac{1}{2}(s+\rho-\ell)$, so that, for example, $\alpha+\beta = s+\rho$. Then

$$\hat{\Psi}(y;\phi_{\ell,s}) = \int_S e(\operatorname{tr}(xy))\,\det(1+ix)^{-\alpha}\,\det(1-ix)^{-\beta}\,dx,$$

and, following the standard manipulations on pp. 274-5 of [23], we have

$$\begin{aligned}
\hat{\Psi}(y;\pi(m(a))\phi_{\ell,s}) &= \det(a)^{s-\rho}\int_S e(\operatorname{tr}(xa^{-1}y\,{}^ta^{-1}))\,\det(1+ix)^{-\alpha}\,\det(1-ix)^{-\beta}\,dx\\
&= \det(a)^{s-\rho}\int_S e(-\operatorname{tr}(xa^{-1}y\,{}^ta^{-1}))\,\det(1-ix)^{-\alpha}\,\det(1+ix)^{-\beta}\,dx\\
&= i^{n(\alpha-\beta)}\,\det(a)^{s-\rho}\,\xi(1,h;\alpha,\beta)\\
&= \det(a)^{s-\rho}\,(2\pi)^{n\rho}\,2^{-n(\rho-1)}\,\Gamma_n(\alpha)^{-1}\Gamma_n(\beta)^{-1}\\
&\qquad\times\int_{u>0,u>2\pi h} e^{2\operatorname{tr}(\pi h-u)}\,\det(u)^{\alpha-\rho}\,\det(u-2\pi h)^{\beta-\rho}\,du\\
&= \det(a)^{s-\rho}\,(2\pi)^{n\rho}\,2^{-n(\rho-1)}\,\Gamma_n(\alpha)^{-1}\Gamma_n(\beta)^{-1}\,\eta(2,\pi h;\alpha,\beta),
\end{aligned}$$

where $h = a^{-1}y\,{}^ta^{-1}$, cf. the top of p. 275 of [23]. Here η is the function defined in (1.26) of loc.cit. Recalling (3.1) of loc. cit., for any $a' \in \mathrm{GL}_n(\mathbb{R})^+$,

$$\eta(g,a'h\,{}^ta';\alpha,\beta) = \det(a')^{2s}\,\eta({}^ta'ga',h;\alpha,\beta),$$

and writing $\pi y = c\epsilon_{p,q}{}^tc$ so that $\pi h = a^{-1}c\epsilon_{p,q}{}^tc\,{}^ta^{-1}$, we have

$$\eta(2,\pi a^{-1}y\,{}^ta^{-1};\alpha,\beta) = \det(a)^{-2s}\,|\det(\pi y)|^s\eta(2\,{}^tc\,{}^ta^{-1}a^{-1}c,\epsilon_{p,q};\alpha,\beta).$$

Next recall that Shimura writes, p. 288,

$$\eta(g,\epsilon_{p,q};\alpha,\beta) = 2^{n(\alpha+\beta-\rho)}\zeta_{p,q}(2g;\alpha,\beta),$$

where

$$\zeta_{p,q}(g;\alpha,\beta) = e^{-\frac{1}{2}\operatorname{tr}(g)}\int_{\substack{x+\epsilon_p>0\\ x+\epsilon'_q>0}} e^{-\operatorname{tr}(gx)}\det(x+\epsilon_p)^{\alpha-\rho}\det(x+\epsilon'_q)^{\beta-\rho}\,dx.$$

Altogether this gives the claimed expression. □

Now we return to the integral (4.8), using the expression just given for $\hat{\Psi}(y;\pi(m(a))\phi_{\ell,s})$. If we substitute the series expansion for $\mathrm{GW}_s(2\pi\epsilon y)$ and switch the order of integration, we obtain the expression

$$\begin{aligned}\omega_1^\epsilon(\pi(m(a))\phi_{\ell,s})_{p,q} = {} & \frac{(2\pi)^{n\rho}\,2^{-n(\rho-1)}}{\Gamma_n(\alpha)\Gamma_n(\beta)}\ \det(v)^{-\frac{1}{2}(s+\rho)}\,(2\pi)^{ns}\,2^{ns} \\ & \times \int\limits_{\substack{x+\epsilon_p>0\\ x+\epsilon_q'>0}} \det(x+\epsilon_p)^{\alpha-\rho}\det(x+\epsilon_q')^{\beta-\rho} \\ & \times \Bigg(\sum_{\mathbf{m}\geq 0}\frac{d_{\mathbf{m}}(-2\pi\epsilon)^{|\mathbf{m}|}}{(\rho)_{\mathbf{m}}\,(s+\rho)_{\mathbf{m}}} \\ & \times \int\limits_{S_{p,q}} e^{-2\pi\mathrm{tr}({}^tcv^{-1}c\,(1+2x))}\,|\det(y)|^s\,\Phi_{\mathbf{m}}(y)\,dy\Bigg)\,dx.\end{aligned} \tag{4.10}$$

Before proceeding, we observe that, in the expression $y = c\epsilon_{p,q}{}^tc$, there is an ambiguity in the choice of c, i.e., only the coset $c\,O(p,q)$ is well defined. More precisely, we have the following basic structural observations where, in particular, (a) implies that the ambiguity in the choice of c has no effect *on the double integral*. The value of the inner integral *does* depend on the choice of c, however!

Lemma 4.6. *(a) There is a bijection*

$$\begin{gathered} X_{p,q} = \{x\in S \mid x+\epsilon_p>0,\ x+\epsilon_q'>0\ \}\\ \downarrow \\ Z_{p,q} = \{z\in S \mid z+\epsilon_{p,q}>0,\ z-\epsilon_{p,q}>0\ \}\end{gathered}$$

given by $x\mapsto 2x+1=z$. The action of $G=\mathrm{GL}_n(\mathbb{R})$ on S induces an action of the group $O(p,q)$ on $Z_{p,q}$. Since $z+\epsilon_{p,q}=2(x+\epsilon_p)$ and $z-\epsilon_{p,q}=2(x+\epsilon_q')$, the quantities $\det(x+\epsilon_p)$ and $\det(x+e_q')$ are constant on the $O(p,q)$-orbits in $Z_{p,q}$.
(b) Let

$$W_{p,q} = \{w\in S_{p,q}\mid 1-w>0, w+1>0\} = S_{p,q}\cap S_{(-1,1)},$$

where

$$S_{(-1,1)} = \{x\in S\mid 1-x>0 \text{ and } 1+x>0\,\}.$$

The action of $G=\mathrm{GL}_n(\mathbb{R})$ on S induces an action of the group $O(n)$ on $W_{p,q}$. Moreover, there is a bijection on orbits

$$O(p,q)\backslash Z_{p,q}\longleftrightarrow O(n)\backslash W_{p,q}$$

defined as follows. For $z\in Z_{p,q}$, write $z=\zeta^t\zeta$ and let $w=\zeta^{-1}\epsilon_{p,q}{}^t\zeta^{-1}$. Then $w\in W_{p,q}$, w depends only on the $O(p,q)$-orbit of z, and the $O(n)$-orbit of w is independent of the choice of ζ. Conversely, for $w\in W_{p,q}$, write $w=\eta\epsilon_{p,q}{}^t\eta$ and let $z=\eta^{-1t}\eta^{-1}$. Then $z\in Z_{p,q}$, z depends only on the $O(n)$-orbit of w, and the $O(p,q)$-orbit of z is independent of the choice of η.

Proof. First note that

$$x+\epsilon_p+x+\epsilon_q' = 2x+1,$$

so that $2x+1$ is automatically positive definite. This also follows from the given conditions on z, viz

$$z + \epsilon_{p,q} + z - \epsilon_{p,q} = 2z > 0.$$

Now, given $x \in X_{p,q}$, we have

$$2x + 1 + \epsilon_{p,q} = 2(x + \epsilon_p) > 0, \qquad 2x + 1 - \epsilon_{p,q} = 2(x + \epsilon'_q) > 0,$$

so that $2x+1$ lies in $Z_{p,q}$. Conversely, if $z \in Z_{p,q}$, then

$$\frac{1}{2}(z-1) + \epsilon_p = \frac{1}{2}(z + \epsilon_{p,q}) > 0, \qquad \frac{1}{2}(z-1) + \epsilon'_q = \frac{1}{2}(z - \epsilon_{p,q}) > 0,$$

so that $\frac{1}{2}(z-1)$ lies in $X_{p,q}$. This proves (a).
To prove (b), we first note that $w \in S_{p,q}$, by construction. We have

$$1 \pm w = 1 \pm \zeta^{-1}\epsilon_{p,q}\zeta^{-1} = \zeta^{-1}(\zeta^t\zeta \pm \epsilon_{p,q})t\zeta^{-1} = \zeta^{-1}(z \pm \epsilon_{p,q})^t\zeta^{-1} > 0,$$

so that $w \in S_{p,q} \cap S_{(-1,1)}$, as claimed. The other direction is analogous. □

Remark 4.7. *It will be useful to note that, under the bijection of part (b),*

$$2^n \det(x + \epsilon_p) = \det(z + \epsilon_{p,q}),$$
$$2^n \det(x + \epsilon'_q) = \det(z - \epsilon_{p,q}),$$

and

$$\det(z \pm \epsilon_{p,q}) = |\det(w)|^{-1} \det(1 \pm w).$$

We can make one simplification in the inner integral in (4.10) as follows. Writing $v = a^t a = ak^t k^t a$ with $k \in O(n)$ and setting $z = 1 + 2x$, as in (a) of the previous lemma, we have

$$\int_{S_{p,q}} e^{-2\pi \mathrm{tr}({}^t c^t a^{-1} a^{-1} c\, z)} |\det(y)|^s \, \Phi_{\mathbf{m}}(y)\, dy \tag{4.11}$$
$$= \det(a)^{2(s+\rho)} \int_{S_{p,q}} e^{-2\pi \mathrm{tr}({}^t cc\, z)} |\det(y)|^s \, \Phi_{\mathbf{m}}(ak \cdot y)\, dy.$$

Since the whole expression is independent of k, integrating over $O(n)$ has no effect, but bringing the $O(n)$ integration inside the $S_{p,q}$-integral, we have

$$\int_{O(n)} \Phi_{\mathbf{m}}(ak \cdot y)\, dk = \Phi_{\mathbf{m}}(a \cdot 1_n)\, \Phi_{\mathbf{m}}(y) = \Phi_{\mathbf{m}}(v)\, \Phi_{\mathbf{m}}(y),$$

by Corollary XI.3.2 in [12]. Thus (4.11) is equal to

$$\det(v)^{s+\rho}\, \Phi_{\mathbf{m}}(v) \int_{S_{p,q}} e^{-2\pi \mathrm{tr}({}^t cc\, z)} |\det(y)|^s \, \Phi_{\mathbf{m}}(y)\, dy.$$

Noting that $dz = 2^{n\rho}\,dx$, for (4.10) we have

$$\omega_1^{\epsilon}(\pi(m(a))\phi_{\ell,s})_{p,q} = \frac{(2\pi)^{n\rho}\,2^{-n(\rho-1)}}{\Gamma_n(\alpha)\Gamma_n(\beta)}\,\det(v)^{\frac{1}{2}(s+\rho)}\,(2\pi)^{ns}$$

$$\times\int\limits_{Z_{p,q}}\det(z+\epsilon_{p,q})^{\alpha-\rho}\det(z-\epsilon_{p,q})^{\beta-\rho} \tag{4.12}$$

$$\Bigg(\times\sum_{\mathbf{m}\geq 0}\frac{d_{\mathbf{m}}(-2\pi\epsilon)^{|\mathbf{m}|}}{(\rho)_{\mathbf{m}}\,(s+\rho)_{\mathbf{m}}}\,\Phi_{\mathbf{m}}(v)$$

$$\times\int\limits_{S_{p,q}}e^{-2\pi\mathrm{tr}({}^tcc\,z)}\,|\det(y)|^{s}\,\Phi_{\mathbf{m}}(y)\,dy\Bigg)\,dz.$$

We need some additional structural information. Let

$$R^+_{p,q} = \{\delta = \mathrm{diag}(\delta_1,\ldots,\delta_n)\mid \delta_1>\cdots>\delta_p>0, \delta_n>\cdots>\delta_{p+1}>0\}.$$

There is then a map

$$O(n)\times R^+_{p,q}\longrightarrow S_{p,q},\qquad (u,\delta)\mapsto u\cdot\delta\epsilon_{p,q} = u\delta\epsilon_{p,q}{}^tu = y \tag{4.13}$$

with open dense image, and, by TheoremVI.2.3 of [12],

$$dy = \Xi_{p,q}(\delta)\,d\delta\,du$$

where

$$\Xi_{p,q}(\delta) = c_{00}\prod_{\substack{1\leq i<j\leq p\\ \text{or}\\ p<i<j\leq n}}|\delta_i-\delta_j|\prod_{i\leq p<j}(\delta_i+\delta_j),$$

and

$$d\delta = d\delta_1\,\ldots\,d\delta_n.$$

Here c_{00} is a certain positive constant depending only on n. Note that the map (4.13) is 2^n to 1, due to the fact that the stabilizer in $O(n)$ of an element $\delta\epsilon_{p,q}$ is the diagonal subgroup, isomorphic to $(\mu_2)^n$.

Let

$$A^+_{p,q} = \{\delta^{\frac{1}{2}}\mid\delta\in R^+_{p,q}\}.$$

Then we have a map

$$O(n)\times A^+_{p,q}\times O(p,q)\longrightarrow G,\qquad (u,a,h)\mapsto uah = g$$

with open dense image, and a left invariant measure dg on G has pullback

$$dg = \det(g)^{-2\rho}\,\Xi_{p,q}(\delta)\,d\delta\,du\,dh, \tag{4.14}$$

where $a^2=\delta$ and dh is a Haar measure on $H = O(p,q)$.

Let

$$R^{+0}_{p,q} = \{\delta\in R^+_{p,q}\mid \delta_j\in(0,1)\text{ for all } j\ \}.$$

Then (4.13) restricts to a map

$$O(n)\times R^{+0}_{p,q}\longrightarrow W_{p,q}$$

with open dense image and we have an injection $R^{+0}_{p,q} \hookrightarrow O(n)\backslash W_{p,q}$. Similarly, we have a map

$$O(p,q) \times R^{+0}_{p,q} \longrightarrow Z_{p,q}, \qquad (h,\delta) \mapsto h \cdot \delta^{-1} = h\,\delta^{-1\,t}h,$$

with open dense image and an injection $R^{+0}_{p,q} \hookrightarrow O(p,q)\backslash Z_{p,q}$.

We now return to (4.12) and write $z = h \cdot \delta_w^{-1}$ with $h \in O(p,q)$ so that

$$dz = \Xi_{p,q}(\delta_w^{-1})\, d(\delta_w^{-1})\, dh = \det(\delta_w)^{-2\rho}\, \Xi_{p,q}(\delta_w)\, d\delta_w\, dh.$$

The double integral becomes

$$\int\limits_{R^{+0}_{p,q}} \det(\delta_w^{-1} + \epsilon_{p,q})^{\alpha-\rho} \det(\delta_w^{-1} - \epsilon_{p,q})^{\beta-\rho}\, \det(\delta_w)^{-2\rho}\, \Xi_{p,q}(\delta_w)$$
$$\times \int\limits_{O(p,q)} \int\limits_{S_{p,q}} e^{-2\pi \mathrm{tr}(ch\cdot\delta_w^{-1})}\, |\det(y)|^s\, \Phi_{\mathbf{m}}(c\cdot\epsilon_{p,q}, 1_n)\, dy\; d\delta_w\, dh.$$

Writing $g = ch$, we have $dy\, dh = (\det g)^{2\rho}\, dg$ and this becomes

$$\int\limits_{R^{+0}_{p,q}} \det(\delta_w^{-1} + \epsilon_{p,q})^{\alpha-\rho} \det(\delta_w^{-1} - \epsilon_{p,q})^{\beta-\rho}\, \det(\delta_w)^{-2\rho}\, \Xi_{p,q}(\delta_w)$$
$$\times \int\limits_{G} e^{-2\pi \mathrm{tr}(g\cdot\delta_w^{-1})}\, |\det(g)|^{2s}\, \Phi_{\mathbf{m}}(g\cdot\epsilon_{p,q}, 1_n)\, (\det g)^{2\rho}\, dg\; d\delta_w.$$

Now we put $g\delta_w^{\frac{1}{2}}$ for g and have

$$\int\limits_{R^{+0}_{p,q}} \det(\delta_w^{-1} + \epsilon_{p,q})^{\alpha-\rho} \det(\delta_w^{-1} - \epsilon_{p,q})^{\beta-\rho}\, \det(\delta_w)^{s-\rho}\, \Xi_{p,q}(\delta_w)$$
$$\times \int\limits_{G} e^{-2\pi \mathrm{tr}(g\,{}^t g)}\, |\det(g)|^{2s}\, \Phi_{\mathbf{m}}(g\cdot\delta_w\epsilon_{p,q}, 1_n)\, (\det g)^{2\rho}\, dg\; d\delta_w.$$

This is

$$\int\limits_{R^{+0}_{p,q}} \det(\delta_w^{-1} + \epsilon_{p,q})^{\alpha-\rho} \det(\delta_w^{-1} - \epsilon_{p,q})^{\beta-\rho}\, \det(\delta_w)^{s-\rho}\, \Xi_{p,q}(\delta_w)$$
$$\times \int\limits_{G} e^{-2\pi \mathrm{tr}({}^t g g)}\, |\det(g)|^{2s}\, \Phi_{\mathbf{m}}(\delta_w\epsilon_{p,q}, {}^t g g)\, (\det g)^{2\rho}\, dg\; d\delta_w,$$

and so, setting $y^\vee = {}^t g g$, we arrive at the expression

$$\int\limits_{R^{+0}_{p,q}} \det(\delta_w^{-1} + \epsilon_{p,q})^{\alpha-\rho} \det(\delta_w^{-1} - \epsilon_{p,q})^{\beta-\rho}\, \det(\delta_w)^{s-\rho}\, \Xi_{p,q}(\delta_w)$$
$$\times \int\limits_{S_{n,0}} e^{-2\pi \mathrm{tr}(y^\vee)}\, \det(y^\vee)^s\, \Phi_{\mathbf{m}}(\delta_w\epsilon_{p,q}, y^\vee)\, dy^\vee\; d\delta_w.$$

By the inversion formula, cf. Lemma XI.2.3 of [12], we have

$$\int_{S_{n,0}} e^{-2\pi \mathrm{tr}(y^\vee)}\ \det(y^\vee)^s\, \Phi_{\mathbf{m}}(\delta_w\epsilon_{p,q}, y^\vee)\, dy^\vee$$
$$= \Gamma_{\mathbf{m}}(s+\rho)\,(2\pi)^{-n(s+\rho)}(2\pi)^{-|\mathbf{m}|}\,\Phi_{\mathbf{m}}(\delta_w\epsilon_{p,q}, 1_n).$$

Returning to the inner sum in (4.12) and canceling the gamma factor, we have

$$\Gamma_n(s+\rho)\,(2\pi)^{-n(s+\rho)}\sum_{\mathbf{m}\ge 0}\frac{(-\epsilon)^{|\mathbf{m}|}d_{\mathbf{m}}}{(\rho)_{\mathbf{m}}}\,\Phi_{\mathbf{m}}(v)\,\Phi_{\mathbf{m}}(\delta_w\epsilon_{p,q}). \tag{4.15}$$

Now, for $v = a^t a$, we write

$$\Phi_{\mathbf{m}}(v)\,\Phi_{\mathbf{m}}(\delta_w\epsilon_{p,q}) = \int_{O(n)} \Phi_{\mathbf{m}}(ak\cdot\delta_w\epsilon_{p,q})\,dk,$$

so that (4.15) becomes the integral over $O(n)$ of

$$\Gamma_n(s+\rho)\,(2\pi)^{-n(s+\rho)}\sum_{\mathbf{m}\ge 0}\frac{(-\epsilon)^{|\mathbf{m}|}d_{\mathbf{m}}}{(\rho)_{\mathbf{m}}}\,\Phi_{\mathbf{m}}(ak\cdot\delta_w\epsilon_{p,q})$$
$$= \Gamma_n(s+\rho)\,(2\pi)^{-n(s+\rho)}\exp(-\epsilon\mathrm{tr}(ak\cdot\delta_w\epsilon_{p,q})),$$

via the standard expansion of $\exp(\mathrm{tr}(z))$, Proposition XII.1.3 (i) of [12]. But now the last three lines of (4.12) amount to

$$\Gamma_n(s+\rho)\,(2\pi)^{-n(s+\rho)}\int_{O(n)}\int_{R^{+0}_{p,q}} \det(\delta_w^{-1}+\epsilon_{p,q})^{\alpha-\rho}\det(\delta_w^{-1}-\epsilon_{p,q})^{\beta-\rho}\ \exp(-\epsilon\mathrm{tr}(ak\cdot\delta_w\epsilon_{p,q}))$$
$$\times\det(\delta_w)^{s-\rho}\,\Xi_{p,q}(\delta_w)\ dk\ d\delta_w$$
$$=\Gamma_n(s+\rho)\,(2\pi)^{-n(s+\rho)}\int_{W_{p,q}} |\det(w)|^{-(\alpha+\beta-2\rho)}\det(1+w)^{\alpha-\rho}\det(1-w)^{\beta-\rho}\ \exp(-\epsilon\mathrm{tr}(a\cdot w))$$
$$\times|\det(w)|^{s-\rho}\,dw.$$

Here the exponent of $|\det(w)|$ is

$$2\rho-\alpha-\beta+s-\rho = 0. \qquad (!!!)$$

Taking a with $a = {}^t a$, we get simply

$$\Gamma_n(s+\rho)\,(2\pi)^{-n(s+\rho)}\int_{W_{p,q}} \det(1+w)^{\alpha-\rho}\det(1-w)^{\beta-\rho}\ \exp(-\epsilon\mathrm{tr}(vw))\,dw,$$

and, altogether:

$$\omega_1^\epsilon(\pi(m(a))\phi_{\ell,s})_{p,q} = B_n(\alpha,\beta)^{-1}\,2^{-n(\rho-1)}\ \det(v)^{\frac{1}{2}(s+\rho)}$$
$$\times\int_{W_{p,q}} \det(1+w)^{\alpha-\rho}\det(1-w)^{\beta-\rho}\ \exp(-\epsilon\mathrm{tr}(vw))\,dw. \tag{4.16}$$

Summing over the signatures, we have

$$\omega_1^\epsilon(\pi(m(a))\phi_{s,\ell}) = B_n(\alpha,\beta)^{-1}\, 2^{-n(\rho-1)}\, \det(v)^{\frac{1}{2}(s+\rho)} \times \int_{1\pm w>0} \det(1+w)^{\alpha-\rho} \det(1-w)^{\beta-\rho} \exp(-\epsilon \mathrm{tr}(vw))\, dw.$$

In the integral here we put $2r = 1+w$ and obtain

$$2^{ns} \exp(\epsilon \mathrm{tr}(v)) \int_{\substack{r>0\\ 1-r>0}} \exp(-2\epsilon \mathrm{tr}(vr))\, \det(r)^{\alpha-\rho} \det(1-r)^{\beta-\rho}\, dr.$$

For $\epsilon = -1$, this gives

$$\omega_1^\epsilon(\pi(m(a))\phi_{s,\ell}) = 2^{n(s-\rho+1)}\, \det(v)^{\frac{1}{2}(s+\rho)} \exp(-\mathrm{tr}(v))\, M_n(\alpha, \alpha+\beta; 2v).$$

and for $\epsilon = +1$, this is

$$2^{ns} \exp(-\mathrm{tr}(v)) \int_{\substack{r>0\\ 1-r>0}} \exp(\mathrm{tr}(2v(1-r))\, \det(r)^{\alpha-\rho} \det(1-r)^{\beta-\rho}\, dr,$$

and hence

$$\omega_1^\epsilon(\pi(m(a))\phi_{s,\ell}) = 2^{n(s-\rho+1)}\, \det(v)^{\frac{1}{2}(s+\rho)} \exp(-\mathrm{tr}(v))\, M_n(\beta, \alpha+\beta; 2v).$$

Finally, taking into account the scaling transformation used to eliminate T, we obtain the claimed expression. This completes the proof of Theorem 4.1. □

5. The ξ-operator

In this section, we construct the ξ-operator, analogous to that defined in [7] and [6], in our present situation. This operator is a slight modification of the $\bar\partial$-operator and is best expressed in terms of differential forms and the Hodge $*$-operator for homogeneous vector bundles on the Siegel space $\mathfrak{H}_n$. Here, as before, we write $\tau = u + iv$, $v > 0$, for an element of $\mathfrak{H}_n$.

5.1. **Homogeneous bundles and differential forms.** For a representation $(\mu, \mathcal{V}_\mu)$ of K, let $\mathcal{L}_\mu$ be the homogeneous vector bundle

$$\mathcal{L}_\mu = (G \times \mathcal{V}_\mu)/K \longrightarrow G/K = \mathfrak{H}_n.$$

Here K acts by $(g, v)\cdot k = (gk, \mu(k)^{-1}v)$, and the C^∞-sections are given by

$$\Gamma(\mathfrak{H}_n, \mathcal{L}_\mu) \simeq [C^\infty(G)\otimes \mathcal{V}_\mu]^K, \qquad \mu(k)\phi(gk) = \phi(g). \tag{5.1}$$

If Γ is a discrete subgroup[6] of $\mathrm{Sp}_n(\mathbb{R})$, we use the same notation for the quotient bundle[7] on $X = \Gamma\backslash\mathfrak{H}_n$. We write $\mathcal{L}_r = \mathcal{L}_{\det^{-r}}$ with $\mathcal{V}_{\det^{-r}} = \mathbb{C}(-r)$, so that sections of $\mathcal{L}_r$ satisfy

[6]The main cases of interest will be $\Gamma \subset \mathrm{Sp}_n(\mathbb{Z})$ an arithmetic subgroup, the intersection of such a subgroup with N, or the trivial subgroup.

[7]Here we should use orbifolds/stacks.

$\phi(gk) = \det(\mathbf{k})^r\, \phi(g)$. Note that the function $j(g,i)^{-r}$ defines a smooth section of $\mathcal{L}_r$ on $\mathfrak{H}_n$. If ϕ is any section, we can write

$$\phi(g) = j(g,i)^{-r} f(\tau),$$

where $f(g(i)) = j(g,i)^r\, \phi(g)$, and invariance of ϕ under left multiplication by an element $\gamma \in \Gamma$ is equivalent to the invariance of f under the corresponding weight r slash operator for γ. The Petersson metric on $\mathcal{L}_r$ is given by $|\phi(g)|^2 = |f(\tau)|^2 \det(v)^r$. More generally, suppose that the representation μ of K on $\mathcal{V}_\mu$ extends to a representation of $K_{\mathbb{C}} \simeq \mathrm{GL}_n(\mathbb{C})$. Let $J(g,\tau) = c\tau + d$ be the canonical automorphy factor $J : G \times \mathfrak{H}_n \longrightarrow K_{\mathbb{C}}$. Note that

$$J(k,i) = A - iB = \overline{\mathbf{k}} = {}^t\mathbf{k}^{-1}.$$

A general smooth section of $\mathcal{L}_\mu$ on $\mathfrak{H}_n$ can be written as

$$\phi(g) = \mu({}^tJ(g,i))\, f(\tau),$$

where f is a smooth $\mathcal{V}_\mu$-valued function of $\mathfrak{H}_n$. The left invariance of the section ϕ under $\gamma \in G$ is equivalent to the invariance

$$f(\gamma\tau) = \mu({}^tJ(\gamma,\tau)^{-1})\, f(\tau). \tag{5.2}$$

Now suppose, moreover, that $\langle\, , \,\rangle_\mu$ is a hermitian inner product on $\mathcal{V}_\mu$ such that $\mu(a)^* = \mu({}^t\bar{a})$ for all $a \in \mathrm{GL}_n(\mathbb{C})$. Such an inner product is 'admissible' in the terminology of [4], p. 47. Then we can define the Petersson metric on $\mathcal{L}_\mu$ by

$$||\phi(g)||^2_\mu = \langle\, \phi(g), \phi(g)\,\rangle_\mu = \langle\, f(\tau), \mu(v^{-1}) f(\tau)\,\rangle_\mu, \qquad v = \mathrm{Im}(\tau) = \mathrm{Im}(g(i)). \tag{5.3}$$

Since

$$v(\gamma(\tau)) = {}^t(c\tau + d)^{-1}\, v\, (c\bar{\tau} + d)^{-1},$$

the right side of (5.3) is γ-invariant if f satisfies (5.2).

If (λ, F_λ) is a finite dimensional unitary representation of Γ, there is an associated flat bundle $\mathcal{F}_\lambda$ on X defined by

$$\mathcal{F}_\lambda = \Gamma\backslash(\mathfrak{H}_n \times F_\lambda),$$

with hermitian metric given by the norm on F_λ.

The bundle Ω^N of top degree holomorphic differential forms on $\mathfrak{H}_n$ is $\mathcal{L}_{n+1}$. Here $N = \frac{1}{2}n(n+1)$. Writing $\tau = \sum_\alpha \tau_\alpha\, e_\alpha$, we let

$$d\mu(\tau) = \wedge_\alpha d\tau_\alpha$$

and note that

$$d\mu(g(\tau)) = j(g,\tau)^{-2\rho}\, d\mu(\tau), \qquad \rho = \frac{1}{2}(n+1).$$

If ϕ is a section of $\mathcal{L}_{n+1}$, the corresponding section of Ω^N is

$$\phi(g)\, j(g,i)^{2\rho}\, d\mu(\tau) = f(\tau)\, d\mu(\tau).$$

Let $\mathcal{E}^{a,b}$ be the bundle of differential forms of type (a,b) on $\mathfrak{H}_n$. We use the same notation for the corresponding bundle on X. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}_+ + \mathfrak{p}_-$$

be the Harish-Chandra decomposition of $\mathfrak{g} = \mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C}$. Then we have an isomorphism

$$\Gamma(\mathfrak{H}_n, \mathcal{E}^{a,b}) = A^{(a,b)}(\mathfrak{H}_n) \xrightarrow{\sim} [\, C^\infty(G) \otimes \wedge^a(\mathfrak{p}_+^*) \otimes \wedge^b(\mathfrak{p}_-^*)\,]^K. \tag{5.4}$$

More explicit coordinates can be given as follows. Let $S = \mathrm{Sym}_n(\mathbb{R})$ with basis e_α and dual basis[8] $e_\alpha^\vee$ with respect to the trace form. There are isomorphisms

$$p_\pm : S_{\mathbb{C}} \xrightarrow{\sim} \mathfrak{p}_\pm, \qquad p_\pm(X) = \frac{1}{2}\begin{pmatrix} X & \pm iX \\ \pm iX & -X \end{pmatrix}. \tag{5.5}$$

Then we have a basis $L_\alpha = p_-(e_\alpha^\vee)$ for $\mathfrak{p}_-$, and we write $\eta'_\alpha \in \mathfrak{p}_-^*$ for the dual basis. The operator on the right side of (5.4) corresponding to $\bar\partial$ is then

$$\bar\partial = \sum_\alpha p_-(e_\alpha^\vee) \otimes \eta'_\alpha, \tag{5.6}$$

where η'_α acts on $\wedge^\bullet(\mathfrak{p}^*)$ by exterior multiplication.

Suppose that E is any hermitian vector bundle on X, and let $\nu : E \xrightarrow{\sim} E^*$ be the conjugate linear isomorphism determined by the hermitian inner product. Recall that the Hodge $*$-operator gives a conjugate linear operator, [27], Chapter V, Section 2,

$$\bar{*}_E : \mathcal{E}^{a,b} \otimes E \longrightarrow \mathcal{E}^{N-a,N-b} \otimes E^*, \qquad \alpha \otimes h \mapsto (*\,\bar\alpha) \otimes \nu(h).$$

Definition. For a hermitian vector bundle E on X, the ξ-operator is defined as

$$\xi = \xi_E = \bar{*}_E\, \bar\partial : \Gamma(X, \mathcal{E}^{a,b} \otimes E) \;\longrightarrow\; \Gamma(X, \mathcal{E}^{N-a,N-b-1} \otimes E^*). \tag{5.7}$$

If $E = \mathcal{F}_\lambda \otimes \mathcal{L}_\mu$ for a unitary flat bundle $\mathcal{F}_\lambda$ and for $\mathcal{L}_\mu$ with the Petersson metric defined by (5.3), then $E^* \simeq \mathcal{F}_{\lambda^\vee} \otimes \mathcal{L}_{\mu^\vee}$ where $\lambda^\vee$ and $\mu^\vee$ are the contragredients of λ and μ.

For example, for an integer κ, a C^∞-section f of the bundle

$$\mathcal{F}_\lambda \otimes \mathcal{E}^{0,N-1} \otimes \mathcal{L}_{n+1-\kappa}. \tag{5.8}$$

can be viewed as an $\mathcal{F}_\lambda$-valued $(0, N-1)$-form on $\mathfrak{H}_n$ of weight $n+1-\kappa$. Then $\xi(f)$ is a section of

$$\mathcal{F}_{\lambda^\vee} \otimes \mathcal{E}^{N,0} \otimes \mathcal{L}_{\kappa-n-1} \;\simeq\; \mathcal{F}_{\lambda^\vee} \otimes \mathcal{L}_\kappa.$$

For $n = 1$, this coincides with the ξ-operator defined in [7].

From now on, to simplify things slightly, we will omit the flat bundle $\mathcal{F}_\lambda$.

[8] Recall that we take e_{jj}, $1 \le j \le n$ and $e_{ij} + e_{ji}$, $1 \le i < j \le n$, as basis for S with dual basis is e_{jj} and $\frac{1}{2}(e_{ij} + e_{ji})$.

It is useful to note that we have the diagram

$$\begin{array}{ccc} \Gamma(X,\mathcal{E}^{a,b}\otimes\mathcal{L}_\mu) & \longrightarrow & [\,C^\infty(\Gamma\backslash G)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^b(\mathfrak{p}_-^*)\,]^K \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \bar\partial} \\ & & [\,C^\infty(\Gamma\backslash G)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^{b+1}(\mathfrak{p}_-^*)\,]^K \\ & & \Big\downarrow{\scriptstyle \bar *} \\ \Gamma(X,\mathcal{E}^{N-a,N-b-1}\otimes\mathcal{L}_{\mu^\vee}) & \longrightarrow & [\,C^\infty(\Gamma\backslash G)\otimes\mathcal{V}_{\mu^\vee}\otimes\wedge^{N-a}(\mathfrak{p}_+^*)\otimes\wedge^{N-b-1}(\mathfrak{p}_-^*)\,]^K \end{array}$$

and that the two maps on the right are given explicitly by (5.6) and

$$\bar * : \phi\otimes x\otimes\omega \longmapsto \bar\phi\otimes\nu(x)\otimes *\bar\omega, \tag{5.9}$$

where $\nu : \mathcal{V}_\mu \xrightarrow{\sim} \mathcal{V}_{\mu^\vee}$ is the conjugate linear isomorphism determined by $\langle\, ,\,\rangle_\mu$.

5.2. **Whittaker forms.** As explained earlier, we consider a version of these operators involving Whittaker forms.

For $T\in\mathrm{Sym}_n(\mathbb{R})$, recall that $\mathcal{W}^T(G)$ is the space of smooth functions ϕ on G such that $\phi(n(b)g) = e(\mathrm{tr}(Tb))\,\phi(g)$. Define the space of Whittaker forms valued in $\mathcal{L}_\mu$ as

$$\mathcal{W}^{-T}(\mathcal{E}^{a,b}\otimes\mathcal{L}_\mu) \xrightarrow{\sim} [\,\mathcal{W}^{-T}(G)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^b(\mathfrak{p}_-^*)\,]^K. \tag{5.10}$$

There is a corresponding ξ-operator described by the diagram

$$\begin{array}{ccc} \mathcal{W}^{-T}(\mathcal{E}^{a,b}\otimes\mathcal{L}_\mu) & \longrightarrow & [\,\mathcal{W}^{-T}(G)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^b(\mathfrak{p}_-^*)\,]^K \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \bar\partial} \\ & & [\,\mathcal{W}^{-T}(G)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^{b+1}(\mathfrak{p}_-^*)\,]^K \\ & & \Big\downarrow{\scriptstyle \bar *} \\ \mathcal{W}^{T}(\mathcal{E}^{N-a,N-b-1}\otimes\mathcal{L}_{\mu^\vee}) & \longrightarrow & [\,\mathcal{W}^{T}(G)\otimes\mathcal{V}_{\mu^\vee}\otimes\wedge^{N-a}(\mathfrak{p}_+^*)\otimes\wedge^{N-b-1}(\mathfrak{p}_-^*)\,]^K, \end{array}$$

where the maps in the right column are given by (5.6) and (5.9).

Our Whittaker functionals (3.17) provide a supply of elements in these spaces via the diagram

$$\begin{array}{ccccc} & \big[\,I(s)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^b(\mathfrak{p}_-^*)\,\big]^K & \xrightarrow{\ \omega_s^{-T}\ } & \big[\,\mathcal{W}^{-T}(G)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^b(\mathfrak{p}_-^*)\,\big]^K & \\ & \Big\downarrow{\scriptstyle \bar\partial} & & \Big\downarrow{\scriptstyle \bar\partial} & \\ \xi & \big[\,I(s)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^{b+1}(\mathfrak{p}_-^*)\,\big]^K & \xrightarrow{\ \omega_s^{-T}\ } & \big[\,\mathcal{W}^{-T}(G)\otimes\mathcal{V}_\mu\otimes\wedge^a(\mathfrak{p}_+^*)\otimes\wedge^{b+1}(\mathfrak{p}_-^*)\,\big]^K & \xi \\ & \Big\downarrow{\scriptstyle \bar *} & & \Big\downarrow{\scriptstyle \bar *} & \\ & \big[\,I(\bar s)\otimes\mathcal{V}_{\mu^\vee}\otimes\wedge^{N-a}(\mathfrak{p}_+^*)\otimes\wedge^{N-b-1}(\mathfrak{p}_-^*)\,\big]^K & \xrightarrow{\ \omega_{\bar s}^{T}\ } & \big[\,\mathcal{W}^{T}(G)\otimes\mathcal{V}_{\mu^\vee}\otimes\wedge^{N-a}(\mathfrak{p}_+^*)\otimes\wedge^{N-b-1}(\mathfrak{p}_-^*)\,\big]^K. & \end{array}$$

Here the maps in the left column are again given by (5.6) and (5.9). We can utilize the fact that the K-spectrum of $I(s)$ is multiplicity free to produce various examples.

5.3. **Some particular vectors.** In the construction of Whittaker forms, we will be interested in the following functions in $I(s)$.

The isomorphism (5.5) satisfies

$$\mathrm{Ad}(k)\,p_+(X) = p_+(\mathbf{k}\cdot X), \qquad \mathbf{k}\cdot X = \mathbf{k}X{}^t\mathbf{k}. \tag{5.11}$$

Similarly, $p_-(X) = \overline{p_+(X)}$ and

$$\mathrm{Ad}(k)\,p_-(X) = p_-(\bar{\mathbf{k}}\cdot X) = p_-({}^t\mathbf{k}^{-1}\cdot X).$$

The trace pairing

$$\langle\, p_+(X), p_-(Y)\,\rangle = \mathrm{tr}(XY)$$

is then invariant under the adjoint action of K, so that $\mathfrak{p}_\pm^* \simeq \mathfrak{p}_\mp$ as K-modules. Note that

$$\wedge^N(\mathfrak{p}_+) \xrightarrow{\sim} \mathbb{C}(2\rho)$$

as K-modules, where $N = n\rho = \dim\mathfrak{p}_\pm$.

For the fixed integer κ, let $r = \kappa - n - 1$, and consider the space

$$\big[\; I(s)\otimes\wedge^{N-1}(\mathfrak{p}_-^*)\otimes\mathbb{C}(r)\;\big]^K. \tag{5.12}$$

Fixing a basis vector $\bar\omega$ for $\wedge^N(\mathfrak{p}_-^*)$, we have a pairing

$$\wedge^{N-1}(\mathfrak{p}_-^*)\otimes\mathfrak{p}_-^* \longrightarrow \wedge^N(\mathfrak{p}_-^*) \xrightarrow{\sim} \mathbb{C}(2\rho),$$

and hence an isomorphism

$$\wedge^{N-1}(\mathfrak{p}_-^*) \xrightarrow{\sim} \mathfrak{p}_-\otimes\mathbb{C}(2\rho). \tag{5.13}$$

Thus

$$\sigma^\vee := \wedge^{N-1}(\mathfrak{p}_-^*)\otimes\mathbb{C}(r) \simeq \mathrm{Sym}_n(\mathbb{C})\otimes\mathbb{C}(\kappa).$$

Note that the vector $v_0 = 1_n \in \mathrm{Sym}_n(\mathbb{C})$ is $O(n)$-invariant, and so, via (5.13), we have the standard function

$$\phi_{s,\sigma}(nm(a)k) = \chi(\det(a))\,|\det(a)|^{s+\rho}\,\det(\mathbf{k})^{-\kappa}\,{}^t\mathbf{k}\,\mathbf{k} \tag{5.14}$$

in (5.12). By (2.9), it is characterized by the invariance property

$$\pi(k)\phi_{s,\sigma} = \det(\mathbf{k})^{-\kappa}\,{}^t\mathbf{k}\cdot\phi_{s,\sigma},$$

together with the normalization $\phi_{s,\sigma}(e) = 1_n$.

For generic s, the function $\phi = \phi_{s,\sigma}$ can also be obtained by applying a certain differential operator to $\phi_{s,-\kappa}$. Here we use the conventions described in more detail in Section 3.3. Let e_α be a basis for $S = \mathrm{Sym}_n(\mathbb{R})$ and let $e_\alpha^\vee$ be the dual basis with respect to the trace form. Let

$$D = \sum_\alpha p_+(e_\alpha)\otimes e_\alpha^\vee \;\in\; \mathfrak{p}_+\otimes S \quad\subset\; U(\mathfrak{g})\otimes S. \tag{5.15}$$

This operator has the following invariance property.

Lemma 5.1. *For $k\in K$,*

$$\mathrm{Ad}(k)\,D = {}^t\mathbf{k}\cdot D = {}^t\mathbf{k}\,D\,\mathbf{k}.$$

Proof. We compute using (5.11)

$$\begin{aligned}\mathrm{Ad}(k)\, D &= \sum_\alpha p_+(\mathbf{k}\cdot e_\alpha)\otimes e_\alpha^\vee = \sum_\alpha\sum_\beta \mathrm{tr}((\mathbf{k}\cdot e_\alpha)e_\beta^\vee)\, p_+(e_\beta)\otimes e_\alpha^\vee\\ &= \sum_\beta p_+(e_\beta)\otimes \sum_\alpha \mathrm{tr}(e_\alpha({}^t\mathbf{k}\cdot e_\beta^\vee))\, e_\alpha^\vee = \sum_\beta p_+(e_\beta)\otimes {}^t\mathbf{k}\cdot e_\beta^\vee = {}^t\mathbf{k}\cdot D.\end{aligned}$$

□

Now consider the function $\pi(D)\phi_{s,-\kappa}\in I(s)\otimes \mathrm{Sym}_n(\mathbb{C})$. For $k\in K$, we have

$$\begin{aligned}\pi(k)\pi(D)\phi_{s,-\kappa} &= \pi(\mathrm{Ad}(k)D)\pi(k)\phi_{s,-\kappa}\\ &= \det(\mathbf{k})^{-\kappa}\, {}^t\mathbf{k}\cdot\pi(D)\phi_{s,-\kappa}.\end{aligned}$$

Thus $\pi(D)\phi_{s,-\kappa}$ is a multiple of $\phi_{s,\sigma}$, and it remains to calculate the constant of proportionality.

Lemma 5.2.

$$\pi(D)\phi_{s,-\kappa} = \frac{1}{2}(s+\rho-\kappa)\,\phi_{s,\sigma} = \alpha(s)\,\phi_{s,\sigma}.$$

Proof. An easy calculation shows that

$$p_+(X)\phi_{s,\ell}(e) = \frac{1}{2}(s+\rho+\ell)\,\mathrm{tr}(X).$$

Therefore

$$D\,\phi_{s,-\kappa} = \frac{1}{2}(s+\rho-\kappa)\,1_n.$$

□

In particular, for $s_0=\kappa-\rho$, we have $p_+(X)\,\phi_{s_0,-\kappa}=0$, so that $\phi_{s_0,-\kappa}$ is a highest weight vector. But this was already clear since this vector is the generator of $R(0,m+2)$, i.e., the image in $I(s_0)$ of the Gaussian for the negative definite space of dimension $m+2$, cf. section 2.6.

Thus we have basis vectors $\phi_{s,-\kappa}$ and $\phi_{s,\sigma}$ in the 1-dimensional spaces on the upper left side of the diagram

(5.16)
$$\begin{array}{ccc}
\big[\,I(s)\otimes\mathbb{C}(\kappa)\,\big]^K & \xrightarrow{\ \omega_1^{-T}\ } & \big[\,\mathcal{W}^{-T}(G)\otimes\mathbb{C}(\kappa)\,\big]^K\\
\downarrow D & & \downarrow D\\
\big[\,I(s)\otimes\wedge^{N-1}(\mathfrak{p}_-^*)\otimes\mathbb{C}(r)\,\big]^K & \xrightarrow{\ \omega_1^{-T}\ } & \big[\,\mathcal{W}^{-T}(G)\otimes\wedge^{N-1}(\mathfrak{p}_-^*)\otimes\mathbb{C}(r)\,\big]^K\\
\downarrow \xi & & \downarrow \xi\\
\big[\,I(\bar{s})\otimes\wedge^{N}(\mathfrak{p}_+^*)\otimes\mathbb{C}(-r)\,\big]^K & \xrightarrow{\ \omega_1^{T}\ } & \big[\,\mathcal{W}^{T}(G)\otimes\wedge^{N}(\mathfrak{p}_+^*)\otimes\mathbb{C}(-r)\,\big]^K\\
\downarrow \wr & & \downarrow \wr\\
\big[\,I(\bar{s})\otimes\mathbb{C}(-\kappa)\,\big]^K & \xrightarrow{\ \omega_1^{T}\ } & \big[\,\mathcal{W}^{T}(G)\otimes\mathbb{C}(-\kappa)\,\big]^K.
\end{array}$$

5.4. **Calculation of $\bar{\partial}$ and ξ.** We now compute the image of $\phi_{s,\sigma}$ under the operator ξ on the left side of (5.16).

Proposition 5.3.

$$\bar{\partial}\,\phi_{s,\sigma} = n(s-\rho+\kappa)\,\phi_{s,-\kappa}\cdot d\mu(\bar{\tau}),$$

and

$$\xi(\phi_{s,\sigma}) = n(\bar{s}-\rho+\kappa)\,\phi_{\bar{s},\kappa}.$$

Here we are slightly abusing notation and writing $d\mu(\bar{\tau})$ for the basis element of $\wedge^N(\mathfrak{p}_-^*)$ arising as the restriction of this global form to the tangent space at i.

Proof. First we apply $\bar{\partial}$:

$$[\ I(s)\otimes\wedge^{N-1}(\mathfrak{p}_-^*)\otimes\mathbb{C}(n+1-\kappa)\ ]^K \xrightarrow{\bar{\partial}} [\ I(s)\otimes\wedge^{N}(\mathfrak{p}_-^*)\otimes\mathbb{C}(n+1-\kappa)\ ]^K,$$

noting that both spaces are 1-dimensional. Using (5.6), and (5.15),

$$\bar{\partial}\,\phi_{s,\sigma} = \alpha(s)^{-1}\,\bar{\partial}\cdot D\,\phi_{s,-\kappa} = \alpha(s)^{-1}\sum_\alpha p_-(e_\alpha^\vee)\,p_+(e_\alpha)\,\phi_{s,-\kappa}\ \cdot d\mu(\bar{\tau}).$$

The second order operator occurring here has the following expression in terms of the Casimir operator (2.8). We have

$$\begin{aligned}\sum_\alpha p_+(e_\alpha)\,p_-(e_\alpha^\vee)+p_-(e_\alpha^\vee)\,p_+(e_\alpha) &= 2\sum_\alpha p_-(e_\alpha^\vee)\,p_+(e_\alpha)+[p_+(e_\alpha),p_-(e_\alpha^\vee)]\\ &= \rho\,H+2\sum_\alpha p_-(e_\alpha^\vee)\,p_+(e_\alpha),\end{aligned}$$

where $H=\sum_j H_j$, for H_j as in Subsection 2.4. On the other hand, a short calculation shows that $C_{\mathfrak{k}}$ acts by $\frac{1}{8}\kappa^2$, whereas H acts by $-n\kappa$. Thus we have

$$\begin{aligned}2\sum_\alpha p_-(e_\alpha^\vee)\,p_+(e_\alpha)\,\phi_{s,-\kappa} &= \big(4n(C-C_{\mathfrak{k}})-\rho\,H\big)\,\phi_{s,-\kappa}\\ &= \frac{1}{2}\,n(s^2-\rho^2+2\rho\kappa-\kappa^2)\,\phi_{s,-\kappa}\\ &= \frac{1}{2}\,n(s-\rho+\kappa)(s+\rho-\kappa)\,\phi_{s,-\kappa}\end{aligned}$$

This gives the first identity. Then

$$\bar{*}\,\bar{\partial}\,\phi_{s,\sigma} = n\,(\bar{s}-\rho+\kappa)\,\phi_{\bar{s},\kappa}\cdot d\mu(\tau),$$

so the second identity is immediate. □

5.5. **Vector valued Whittaker functions.** Now we can apply the Whittaker functionals to obtain Whittaker forms on the right side of (5.16). Recall that $f_{s,-\kappa}(g) = \omega_1^{-T}(\pi(m(g))\phi_{s,-\kappa})$ and let

$$\boldsymbol{f}_{s,\sigma}(g) = \omega_1^{-T}(\pi(m(g))\phi_{s,\sigma}). \tag{5.17}$$

Then by Lemma 5.2,

$$\alpha(s)\,\boldsymbol{f}_{s,\sigma}(g) = Df_{s,-\kappa}(g) = \sum_{\alpha} p_+(e_\alpha) f_{s,-\kappa}(g)\, e_\alpha^\vee, \tag{5.18}$$

where $\alpha(s) = \frac{1}{2}(s-s_0) = \frac{1}{2}(s+\rho-\kappa)$. Thus $\boldsymbol{f}_{s,\sigma}$ is S-valued and, by Lemma 5.1, satisfies

$$\boldsymbol{f}_{s,\sigma}(gk) = \det(\mathbf{k})^{-\kappa}\, {}^t\mathbf{k}\, \boldsymbol{f}_{s,\sigma}(g)\, \mathbf{k}. \tag{5.19}$$

Note that the scaling relation

$$\boldsymbol{f}_{s,\sigma}^{-T}(m(c)g) = \det(c)^{s+\rho}\, \boldsymbol{f}_{s,\sigma}^{-cT^tc}(g), \qquad c \in \mathrm{GL}_n(\mathbb{R})^+, \tag{5.20}$$

holds for $\boldsymbol{f}_{s,\sigma} = \boldsymbol{f}_{s,\sigma}^{-T}$, where we include the normally omitted superscript.

As a consequence of our constructions, we have obtained the following.

Theorem 5.4. *The Whittaker forms $\boldsymbol{f}_{s,\sigma}$ defined by (5.18) lie in the space*

$$\big[\,\mathcal{W}^{-T}(G) \otimes \wedge^{N-1}(\mathfrak{p}_-^*) \otimes \mathbb{C}(r)\,\big]^K.$$

(i) The infinitesimal character of $\boldsymbol{f}_{s,\sigma}$ is $\chi_{\lambda(s)+\rho_G}$. In particular, for the Casimir operator C,

$$C\cdot \boldsymbol{f}_{s,\sigma} = \frac{1}{8}\,(s+\rho)(s-\rho)\,\boldsymbol{f}_{s,\sigma}.$$

(ii) For the ξ-operator,

$$\xi(\boldsymbol{f}_{s,\sigma}) = n\,(\bar{s}-\rho+\kappa)\,\overline{f_{s,-\kappa}}$$

(iii)[9] *At $s = s_0 = \kappa-\rho$,*

$$\xi(\boldsymbol{f}_{s_0,\sigma})(g) = c(n,s_0)\,W_\kappa^T(g),$$

where

$$W_\kappa^T(n(b)m(a)k) = \det(\mathbf{k})^\kappa\,\det(a)^\kappa\, e(\mathrm{tr}(T\tau)) = j(g,i)^{-\kappa}\, q^T, \tag{5.21}$$

$v = a^ta$, $\tau = b+iv$, and

$$c(n,s_0) = 2^{n(\kappa-2\rho+1)}.$$

Proof. The first two statements follow from the commutativity of (5.16) and the corresponding results for the vectors on the left side. In particular,

$$\xi(\boldsymbol{f}_{s,\sigma}) = n\,(\bar{s}-\rho+\kappa)\,\overline{f_{s,-\kappa}}.$$

By (4.5), we have

$$\overline{f_{s,-\kappa}}(n(b)m(a)k) = c(n,\bar{s})\,q^T\,\det(v)^{\frac{1}{2}(\bar{s}+\rho)}\,\det(\mathbf{k})^\kappa\,\overline{M_n(\alpha,\alpha+\beta;4\pi cv^tc)}.$$

But, at $s = s_0 = \kappa-\rho$, $\alpha = 0$ and $M_n(0,s_0;4\pi cv^tc) = 1$, so we get the claimed expression. □

[9]Here note that, although $\alpha = \alpha(s_0) = 0$, the calculations of Section 5.6 show that the right side of (5.18) is divisible by α and that $\boldsymbol{f}_{s_0,\sigma}$ is given by a nice convergent power series in v.

5.6. **Some explicit formulas.** In this section, we calculate the function $\boldsymbol{f}_{s,\sigma}$ more explicitly. In view of the scaling relation (5.20), it suffices to consider the case $T = 1_n$. Recall that

$$M_n(\alpha, \alpha+\beta; z) = \sum_{\mathbf{m}\geq 0} \frac{(\alpha)_{\mathbf{m}}}{(s+\rho)_{\mathbf{m}}} \frac{d_{\mathbf{m}}}{(\rho)_{\mathbf{m}}} \Phi_{\mathbf{m}}(z),$$

where $\mathbf{m} = (m_1, \dots, m_n)$ with $m_1 \geq m_2 \geq \cdots \geq m_m \geq 0$. Since

$$(\alpha)_{\mathbf{m}} = (\alpha)_{m_1}(\alpha - \frac{1}{2})_{m_2} \dots (\alpha - \frac{1}{2}(n-1))_{m_n},$$

we have $(\alpha)_0 = 1$ and $(0)_{\mathbf{m}} = 0$ for $\mathbf{m} \neq 0$. In particular, $M_n(0, \beta; z) = 1$. For $\mathbf{m} \neq 0$, let

$$[0]_{\mathbf{m}} = (\alpha^{-1}(\alpha)_{\mathbf{m}}))|_{\alpha=0}.$$

Then

$$[0]_{\mathbf{m}} = \begin{cases} (1)_{m_1-1}(-\frac{1}{2})_{m_2}\,(-\frac{1}{2})^r(r+1)!, & \text{if } \mathbf{m} = (m_1, m_2, 1, \dots, 1, 0, \dots, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Here in the first case $\mathbf{m} > 0$ with $m_3 \leq 1$ and with a string of r 1's following m_2. With this notation, we can state our result.

Proposition 5.5. *(i) For $a \in \mathrm{GL}_n(\mathbb{R})^+$,*

$$\boldsymbol{f}_{s,\sigma}(m(a)) = c(n,s)\, e^{-2\pi \mathrm{tr}(v)}\, \det(v)^{\frac{1}{2}(s+\rho)}\, B_n(\alpha,\beta)^{-1} \times \int\limits_{\substack{t>0 \\ 1-t>0}} \big(\, 1_n + \alpha(s)^{-1}\, 4\pi\, {}^t a\, t\, a \,\big)\, e^{4\pi \mathrm{tr}(tv)}\, \det(t)^{\alpha-\rho} \det(1-t)^{\beta-\rho}\, dt. \tag{5.22}$$

Recall that $c(n,s) = 2^{n(s-\rho+1)}$.

(ii) For $n = 1$,

$$\boldsymbol{f}_{s_0,\sigma}(m(a)) = 2^{\kappa-1}\, 4\pi\, e^{2\pi v}\, v^{\frac{1}{2}\kappa+1} \int_0^1 e^{-4\pi t v}\, t^{\kappa-1}\, dt.$$

(iii) For $n \geq 2$,

$$\boldsymbol{f}_{s_0,\sigma}(m(a)) = 2^{n(\kappa-n)}\, e^{-2\pi \mathrm{tr}(v)}\, \det(v)^{\frac{1}{2}\kappa} \Bigg(\, 1_n + \sum_{\mathbf{m}>0} \frac{[0]_{\mathbf{m}}}{(\kappa)_{\mathbf{m}}} \frac{d_{\mathbf{m}}}{(\rho)_{\mathbf{m}}}\, {}^t a\, \frac{\partial}{\partial v} \{\, \Phi_{\mathbf{m}}(4\pi v)\, \}\, a \,\Bigg).$$

Here $v = a^t a$, as usual.

Remark 5.6. *In the case $n = 1$, we recover the basic formula from [6]. Notice that, in the case $n \geq 2$, we do not yet have a complete evaluation of $\boldsymbol{f}_{s_0,\sigma}(m(a))$; it would be interesting to have a nicer closed formula.*

Proof. Recalling (5.18), we begin by computing some derivatives.

Lemma 5.7. *For $X \in S$,*

$$p_+(X) f_{s,\ell}(m(a)) = \frac{1}{2} \big(\, D_X + \ell\, \mathrm{tr}(X) + 4\pi\, \mathrm{tr}(aX^t a) \,\big) \cdot f_{s,\ell}(m(a)).$$

Here, for a function h on $\mathrm{GL}_n(\mathbb{R})$,

$$D_X h(a) = \frac{d}{dt} \,(\, h(a \exp(tX))\,)\, |_{t=0}.$$

In particular, for a function of $v = a\,{}^t a$,

$$D_X = 2\,\mathrm{tr}(aX^t a\,\frac{\partial}{\partial v}).$$

This follows from a simple direct calculation. Then, recalling the expression of the function $f_{s,\ell}$ given in Theorem 4.1, we have the nice expression for our Whittaker form

$$\begin{aligned}\alpha(s)\,\boldsymbol{f}_{s,\sigma}(m(a)) &= \sum_{\alpha} p_+(e_\alpha) f_{s,-\kappa}(m(a))\,e_\alpha^\vee \\ &= c(n,s)\,e^{-2\pi\mathrm{tr}(v)}\,\det(v)^{\frac{1}{2}(s+\rho)}\,B_n(\alpha,\beta)^{-1} \\ &\quad\times \int_{\substack{t>0\\1-t>0}} \big(\,\alpha(s)\,1_n + 4\pi\,{}^t a\,t\,a\,\big)\,e^{4\pi\mathrm{tr}(tv)}\,\det(t)^{\alpha-\rho}\det(1-t)^{\beta-\rho}\,dt.\end{aligned} \tag{5.23}$$

This gives statement (i).

When $n = 1$, (5.23) amounts to

$$\begin{aligned}\boldsymbol{f}_{s,\sigma}(m(a)) &= c(1,s)\,e^{-2\pi v}\,v^{\frac{1}{2}(s+1)}\,\alpha(s)^{-1}\,B(\alpha,\beta)^{-1} \\ &\quad\times \int_0^1 \big(\,\alpha(s) + 4\pi tv\,\big)\,e^{4\pi tv}\,t^{\alpha-1}\,(1-t)^{\beta-1}\,dt.\end{aligned} \tag{5.24}$$

Lemma 5.8.

$$\alpha^{-1}\,B(\alpha,\beta)^{-1}\int_0^1 \big(\,\alpha(s) + 4\pi tv\,\big)\,e^{4\pi tv}\,t^{\alpha-1}\,(1-t)^{\beta-1}\,dt = M(\alpha+1, s+1; 4\pi v).$$

Proof. Initially, we have

$$\begin{aligned}\alpha^{-1}\,B(\alpha,\beta)^{-1}&\int_0^1 \big(\,\alpha(s) + 4\pi tv\,\big)\,e^{4\pi tv}\,t^{\alpha-1}\,(1-t)^{\beta-1}\,dt \\ &= M(\alpha, s+1; 4\pi v) + \alpha^{-1}\,v\,\frac{\partial}{\partial v}\{M(\alpha, s+1; 4\pi v)\}.\end{aligned}$$

But by 13.4.10 of [1],

$$4\pi v\,M'(\alpha, s+1; 4\pi v) = -\alpha\,M(\alpha, s+1; 4\pi v) + \alpha\,M(\alpha+1, s+1; 4\pi v).$$

□

Thus

$$\boldsymbol{f}_{s,\sigma}(m(a)) = 2^s\,e^{-2\pi v}\,v^{\frac{1}{2}(s+1)}\,M(\alpha+1, s+1; 4\pi v).$$

This agrees with the Whittaker form for $m = 1$ in [6], up to a simple factor. To evaluate at $s = s_0$ we return to the original expression (5.24). Since

$$\alpha^{-1}\,B(\alpha,\beta)^{-1} = \frac{\Gamma(s+1)}{\Gamma(\alpha+1)\Gamma(\beta)},$$

we can plug in $s_0 = \kappa - 1$ so that $\alpha_0 = 0$ and $\beta_0 = \kappa$, and obtain the following expression

$$\begin{aligned}
\boldsymbol{f}_{s_0,\sigma}(g) &= c_0\, e^{-2\pi v}\, v^{\frac{1}{2}\kappa} \int_0^1 \big(\, 4\pi t v\,\big)\, e^{4\pi t v}\, t^{-1}\, (1-t)^{\kappa-1}\, dt \\
&= c_0\, 4\pi\, e^{-2\pi v}\, v^{\frac{1}{2}\kappa+1} \int_0^1 e^{4\pi t v}\, (1-t)^{\kappa-1}\, dt \\
&= c_0\, 4\pi\, e^{2\pi v}\, v^{\frac{1}{2}\kappa+1} \int_0^1 e^{-4\pi t v}\, t^{\kappa-1}\, dt \\
&= c_0\, 4\pi\, e^{2\pi v}\, v^{\frac{1}{2}\kappa+1} \Big[\, (4\pi v)^{-\kappa}\Gamma(\kappa) - \int_1^\infty e^{-4\pi t v}\, t^{\kappa-1}\, dt\, \Big],
\end{aligned}$$

where $c_0 = c(1, s_0) = 2^{\kappa-1}$. The last expression here is in some ways more enlightening, although we do note record it in the statement (ii) of the proposition.

Finally, for general $n \geq 2$, we want to evaluate (5.22) at $s_0 = \kappa - \rho$. The term associated to 1_n is given by the scalar matrix

$$\begin{aligned}
&c(n,s)\, e^{-2\pi \mathrm{tr}(v)}\, \det(v)^{\frac{1}{2}(s+\rho)}\, B_n(\alpha,\beta)^{-1} \\
&\qquad \times \int\limits_{\substack{t>0\\ 1-t>0}} e^{4\pi \mathrm{tr}(tv)}\, \det(t)^{\alpha-\rho} \det(1-t)^{\beta-\rho}\, dt \cdot 1_n \\
&= c(n,s)\, e^{-2\pi \mathrm{tr}(v)}\, \det(v)^{\frac{1}{2}(s+\rho)}\, M_n(\alpha, \alpha+\beta; 4\pi v)\, \cdot 1_n.
\end{aligned}$$

Since $M_n(0, \kappa; z) = 1$, we obtain a contribution

$$2^{n(\kappa-n)}\, e^{-2\pi \mathrm{tr}(v)}\, \det(v)^{\frac{1}{2}\,\kappa}\, \cdot 1_n.$$

The other contribution is the value of

$$\begin{aligned}
&c(n,s)\, e^{-2\pi \mathrm{tr}(v)}\, \det(v)^{\frac{1}{2}(s+\rho)} \\
&\quad \times {}^t a\, \Big(\, \alpha^{-1}\, B_n(\alpha,\beta)^{-1} \int\limits_{\substack{t>0\\ 1-t>0}} 4\pi t\, e^{4\pi \mathrm{tr}(tv)}\, \det(t)^{\alpha-\rho} \det(1-t)^{\beta-\rho}\, dt\, \Big)\, a
\end{aligned}$$

at s_0. The inner integral here is just

$$\alpha^{-1} \frac{\partial}{\partial v} \big\{\, M_n(\alpha, \alpha+\beta; 4\pi v)\, \big\} = \sum_{\mathbf{m}>0} \frac{\alpha^{-1}\, (\alpha)_{\mathbf{m}}}{(s+\rho)_{\mathbf{m}}}\, \frac{d_{\mathbf{m}}}{(\rho)_{\mathbf{m}}}\, \frac{\partial}{\partial v} \big\{\, \Phi_{\mathbf{m}}(4\pi v)\, \big\},$$

where we can omit the term $\mathbf{m} = 0$ since $\Phi_0(z) = 1$ is killed by $\partial/\partial v$. Note that, for $\mathbf{m} \neq 0$, $\alpha^{-1}(\alpha)_{\mathbf{m}}$ is finite at $\alpha = 0$ and vanishes if $m_3 > 1$. With the notation explained above, we arrive at the expression given in (iii). □

6. A global construction

In this section, we define the space of Whittaker forms and discuss the 'global' ξ-operator. For simplicity, we restrict to the case of $\Gamma = \mathrm{Sp}_n(\mathbb{Z})$ and a positive even integer κ.

For $T \in \mathrm{Sym}_n(\mathbb{Z})^{\vee}_{>0}$, we consider the basic Whittaker form

$$\boldsymbol{f}^{-T}_{s,\sigma} \in \big[\,\mathcal{W}^{-T}(G) \otimes \wedge^{N-1}(\mathfrak{p}^*_-) \otimes \mathbb{C}(\kappa - n - 1)\,\big]^K. \tag{6.1}$$

These forms are the analogues of those considered in Section 4 of [6] in the case of $\mathrm{SL}_2(\mathbb{R})$, i.e., $n = 1$. By construction, they are invariant under the translation subgroup Γ^u_∞ of $\mathrm{Sp}_n(\mathbb{Z})$. Setting $s = s_0$, we have Whittaker forms $\boldsymbol{f}^{-T}_{s_0,\sigma}$ satisfying

$$C \cdot \boldsymbol{f}^{-T}_{s_0,\sigma} = \frac{1}{8}\,\kappa(\kappa - n - 1)\,\boldsymbol{f}^{-T}_{s_0,\sigma},$$

where C is the Casimir operator for G.

Let

$$\mathbb{H}_{n+1-\kappa}(G) \ \subset\ \big[C^\infty(G) \otimes \wedge^{N-1}(\mathfrak{p}^*_-) \otimes \mathbb{C}(\kappa - n - 1)\,\big]^K$$

be the subspace spanned by these forms as T varies over $\mathrm{Sym}_n(\mathbb{Z})^{\vee}_{>0}$. Clearly, the basic forms $\boldsymbol{f}^{-T}_{s_0,\sigma}$ are linearly independent. Let $\mathbb{M}_\kappa(G)$ be the subspace of $C^\infty(G)$ spanned by the functions W^κ_T given by (5.21) as T varies over $\mathrm{Sym}_n(\mathbb{Z})^{\vee}_{>0}$. By (iii) of Theorem 5.4, the ξ-operator induces an isomorphism

$$\xi : \mathbb{H}_{n+1-\kappa}(G) \xrightarrow{\ \sim\ } \mathbb{M}_\kappa(G).$$

On the other hand, if $\kappa > 2n$, the classical theory of Poincaré series, [19], implies that the series

$$\mathcal{P}^T_\kappa(g) = \sum_{\Gamma^u_\infty\backslash\Gamma} W^T_\kappa(\gamma g)$$

converges absolutely and uniformly on compact subsets of G. The resulting functions are 'holomorphic' cusp forms and span the space $S_\kappa(\Gamma)$ of such forms. Thus we have constructed a diagram

$$\begin{array}{ccc} \mathbb{H}_{n+1-\kappa}(G) & \xrightarrow[\sim]{\ \xi\ } & \mathbb{M}_\kappa(G) \\ & \searrow^{\xi_\Gamma} & \downarrow{\scriptstyle \mathcal{P}_\Gamma} \\ & & S_\kappa(\Gamma). \end{array} \tag{6.2}$$

For example, assume that $S_\kappa(\Gamma)$ is one-dimensional and let χ be a generator (this is for instance the case for $n = 2$ and $\kappa = 10$ or 12, where $S_\kappa(\Gamma)$ is spanned by the Igusa cusp form χ_{10} or χ_{12}). Then, using the fundamental formula (7) of [19], it is easily seen that

$$\xi_\Gamma(\boldsymbol{f}^{-T}_{s_0,\sigma}) = A \cdot \frac{(\det T)^{\frac{n+1}{2}-\kappa}}{\varepsilon(T)}\overline{a_T(\chi)} \cdot \frac{\chi}{\|\chi\|^2},$$

where A is a non-zero constant independent of T, and $\varepsilon(T)$ is the order of the stabilizer of T in $\mathrm{GL}_n(\mathbb{Z})$. Moreover, $a_T(\chi)$ denotes the T-th Fourier coefficient of χ, and $\|\chi\|$ its Petersson norm.

7. Appendix: Notation

We summarize the slight variation of the notation from [12] and [22] used in this paper. In particular, we specialize to the case of the formally real Jordan algebra $S = \mathrm{Sym}_n(\mathbb{R})$. Here is a list of notation:

$$G = \mathrm{Sp}_n(\mathbb{R})$$
$$S = \mathrm{Sym}_n(\mathbb{R})$$

For $g \in \mathrm{GL}_n(\mathbb{C})$ and $x \in S_{\mathbb{C}} = \mathrm{Sym}_n(\mathbb{C})$, $g \cdot x = gx\,{}^t g$.

$$\rho = \frac{1}{2}(n+1)$$
$$\mathcal{P}(S_{\mathbb{C}}) = \text{polynomial functions}, \quad \ell(g)f(z) = f(g^{-1}\cdot z)$$
$$\Delta_j(z) = \text{principal } j\times j\text{-minor of } z \in S_{\mathbb{C}},\ \Delta_n(z) = \det(z)$$
$$\mathbf{m} = (m_1 \geq m_2 \geq \dots \geq m_n), \quad m_j \in \mathbb{Z}$$

$$\Delta_{\mathbf{m}}(z) = \Delta_1(z)^{m_1-m_2}\dots\Delta_j(z)^{m_{n-1}-m_n}\Delta_n(z)^{m_n}$$
$$\mathcal{P}_{\mathbf{m}}(S_{\mathbb{C}}) = \text{subspace generated by } \mathrm{GL}_n(\mathbb{C})\text{-translates of } \Delta_{\mathbf{m}},\ \mathbf{m}\geq 0$$
$$d_{\mathbf{m}} = \dim_{\mathbb{C}}\mathcal{P}_{\mathbf{m}}(S_{\mathbb{C}})$$
$$\Phi_{\mathbf{m}}(z) = \int_{O(n)}\Delta_{\mathbf{m}}(k\cdot z)\,dk$$
$$\Gamma_n(s) = (2\pi)^{\frac{1}{4}n(n-1)}\,\Gamma(s)\Gamma(s-\frac{1}{2})\dots\Gamma(s-\frac{1}{2}(n-1))$$
$$B_n(\alpha,\beta) = \frac{\Gamma_n(\alpha)\Gamma_n(\beta)}{\Gamma_n(\alpha+\beta)}$$
$$\Gamma_{\mathbf{m}}(\lambda) = \Gamma_n(\lambda+\mathbf{m}) = \prod_{i=1}^{n}\Gamma(\lambda+m_i-\frac{1}{2}(i-1))$$
$$\Gamma_{\mathbf{m}}(\lambda) = (\lambda)_{\mathbf{m}}\,\Gamma_n(\lambda)$$
$$(\lambda)_{\mathbf{m}} = \prod_{i=1}^{n}\big(\lambda-\frac{1}{2}(i-1)\big)_{m_i}$$
$$(s)_m = s(s+1)\dots(s+m-1) = \frac{\Gamma(s+m)}{\Gamma(s)}.$$

Note that the notation $\Gamma_{\mathbf{m}}(\lambda)$ does not seem to be standard, but it is frequently convenient. The function $\Phi_{\mathbf{m}}(z,w)$ on $S_{\mathbb{C}}\times S_{\mathbb{C}}$ defined in [22] Lemma 1.11 is characterized by the following two properties:

(i) $\Phi_{\mathbf{m}}(z,1_n) = \Phi_{\mathbf{m}}(z)$.
(ii) For $a \in \mathrm{GL}_n(\mathbb{C})$,

$$\Phi_{\mathbf{m}}(a\cdot z,w) = \Phi_{\mathbf{m}}(z,{}^t\bar{a}\cdot w).$$

For the trace form $\langle x, y \rangle = \mathrm{tr}(xy)$ on S and the standard basis

$$\{e_\alpha\} = \{e_{ii}, e_{ij} + e_{ji} \mid \ 1 \le i \le n, i < j\}$$

the dual basis is

$$\{e_\alpha^\vee\} = \{e_{ii}, \frac{1}{2}(e_{ij} + e_{ji}) \mid \ 1 \le i \le n, i < j\}.$$

At several points in the calculations, we need the following inversion formula, [12], Chapter XI, Lemma XI.2.3. For $p \in \mathcal{P}_{\mathbf{m}}(S_{\mathbb{C}})$, and $\mathrm{Re}(\lambda) > \rho - 1$,

$$\int_{x>0} e^{-\mathrm{tr}(xy)} p(x)\, \det(x)^{\lambda-\rho}\, dx = \Gamma_n(\lambda + \mathbf{m})\, \det(y)^{-\lambda}\, p(y^{-1}).$$

We also recall that

$$e^{\mathrm{tr}(z\bar{w})} = \sum_{\mathbf{m}\ge 0} \frac{d_{\mathbf{m}}}{(\rho)_{\mathbf{m}}}\, \Phi_{\mathbf{m}}(z, w),$$

and that, for $\lambda \in \mathbb{C}$ with $(\lambda)_{\mathbf{m}} \ne 0$ for all $\mathbf{m} \ge 0$, z, $w \in S_{\mathbb{C}}$ the J-Bessel function is defined by, [22], p.818,

$$\mathcal{J}_\lambda(z, w) = \sum_{\mathbf{m}\ge 0} (-1)^{|\mathbf{m}|} \frac{d_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}\, (\rho)_{\mathbf{m}}}\, \Phi_{\mathbf{m}}(z, w).$$

7.1. **An estimate.** The following classical estimate will be useful.

Lemma 7.1. *(i) For* $\mathrm{Re}(\alpha) > \rho$ *and* $\mathrm{Re}(\beta - \alpha) > \rho$,

$$|{}_1F_1(\alpha, \beta; v)| \le e^{\mathrm{tr}(v)}\, \det(v)^{\mathrm{Re}(\alpha-\beta)} \Gamma_n(\mathrm{Re}(\beta - \alpha))|B_n(\alpha, \beta - \alpha)|^{-1}.$$

(ii) Suppose that α *and* β *are real with* $\alpha > \rho$ *and* $\beta - \alpha > \rho$. *Then for any* ϵ *with* $0 < \epsilon < 1$,

$$|{}_1F_1(\alpha, \beta; v)| \ge C_\epsilon\, e^{(1-\epsilon)\mathrm{tr}(v)},$$

where $C_\epsilon > 0$ *depends on* ϵ, α *and* β *and can be taken uniformly for* α *and* β *in a compact set.*

Proof. Suppose that $v = a^2$ where $a = {}^t a$, and, using (4.3), consider

$$\begin{aligned}
{}_1F_1(\alpha, \beta; v) &= B_n(\alpha, \beta - \alpha)^{-1} \int\limits_{\substack{t>0\\1-t>0}} e^{\mathrm{tr}(vt)}\, \det(t)^{\alpha-\rho}\, \det(1-t)^{\beta-\alpha-\rho}\, dt \\
&= B_n(\alpha, \beta - \alpha)^{-1}\, e^{\mathrm{tr}(v)} \int\limits_{\substack{t>0\\1-t>0}} e^{-\mathrm{tr}(v(1-t))}\, \det(t)^{\alpha-\rho}\, \det(1-t)^{\beta-\alpha-\rho}\, dt \\
&= B_n(\alpha, \beta - \alpha)^{-1}\, e^{\mathrm{tr}(v)} \int\limits_{\substack{r>0\\1-r>0}} e^{-\mathrm{tr}(vr)}\, \det(1-r)^{\alpha-\rho}\, \det(r)^{\beta-\alpha-\rho}\, dr \\
&= B_n(\alpha, \beta - \alpha)^{-1}\, e^{\mathrm{tr}(v)}\, \det(v)^{\alpha-\beta} \int\limits_{\substack{r>0\\v-r>0}} e^{-\mathrm{tr}(r)}\, \det(1 - a^{-1}ra^{-1})^{\alpha-\rho}\, \det(r)^{\beta-\alpha-\rho}\, dr.
\end{aligned}$$

Now $1 \geq a^{-1}ra^{-1} > 0$ so that, for $\mathrm{Re}(\alpha) > \rho$, the factor $\det(1 - a^{-1}ra^{-1})^{\alpha-\rho}$ lies in $(0,1)$ and we have

$$|{}_1F_1(\alpha,\beta;v)| \leq |B_n(\alpha,\beta-\alpha)|^{-1}\, e^{\mathrm{tr}(v)} \det(v)^{\mathrm{Re}(\alpha-\beta)} \int\limits_{\substack{r>0\\ v-r>0}} e^{-\mathrm{tr}(r)} \det(r)^{\mathrm{Re}(\beta-\alpha)-\rho}\, dr.$$

The integral here is bounded by

$$\int\limits_{r>0} e^{-\mathrm{tr}(r)} \det(r)^{\mathrm{Re}(\beta-\alpha)-\rho}\, dr = \Gamma_n(\mathrm{Re}(\beta-\alpha)).$$

For the lower bound, we have

$$\begin{aligned} |{}_1F_1(\alpha,\beta;v)| &\geq |B_n(\alpha,\beta-\alpha)|^{-1}\, e^{\mathrm{tr}(v)} \int\limits_{\substack{r>\frac{1}{2}\epsilon 1_n\\ \epsilon 1_n - r>0}} e^{-\mathrm{tr}(vr)} \det(1-r)^{\alpha-\rho} \det(r)^{\beta-\alpha-\rho}\, dr \\ &\geq |B_n(\alpha,\beta-\alpha)|^{-1}\, e^{(1-\epsilon)\mathrm{tr}(v)} \int\limits_{\substack{r>\frac{1}{2}\epsilon 1_n\\ \epsilon 1_n - r>0}} \det(1-r)^{\alpha-\rho} \det(r)^{\beta-\alpha-\rho}\, dr. \end{aligned}$$

□

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