2 Problems: Quantum mechanics essentials

2.1. Show that if a wavefunction $\psi(x,t)$ satisfies the one-dimensional Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\hbar^2\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$$

for a real potential $V(x)$, and assuming that ψ and $\frac{\partial\psi}{\partial x}$ vanish as $x \to \pm\infty$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} |\psi(x,t)|^2 \,\mathrm{d}x = 0.$$

What does this equation mean?

2.2. Suppose $|\alpha\rangle$ and $|\beta\rangle$ are eigenstates of a self-adjoint operator $\hat{A}$, with eigenvalues α and β respectively. Using the definition of the adjoint of $\hat{A}$ and inner products such as $\left\langle \alpha \middle| \hat{A} \middle| \alpha \right\rangle$ and $\left\langle \alpha \middle| \hat{A} \middle| \beta \right\rangle$, show that:

(a) $\alpha \in \mathbb{R}$ (hence all eigenvalues of a self-adjoint operator are real.)

(b) If $\alpha \neq \beta$ then $\langle \alpha | \beta \rangle = 0$ (hence eigenstates of a self-adjoint operator with *different* eigenvalues are orthogonal.)

2.3. Suppose a self-adjoint operator $\hat{A}$ has normalised eigenstates $|1\rangle$, $|2\rangle$ and $|3\rangle$ with eigenvalues 1, 2, 3 respectively. **PROBLEMS CLASS 1**

(a) What is the probability of measuring $A = 2$ if the system is described by the state:

(i) $|\psi\rangle = \frac{1}{2}|1\rangle + \frac{1}{4}|2\rangle + \frac{1}{4}|3\rangle$?

(ii) $|\phi\rangle = \frac{3}{4}|1\rangle - \frac{1}{2}|2\rangle + \frac{1}{4}|3\rangle$?

(b) Calculate $\langle A\rangle$ for the state $|\psi\rangle$ and for the state $|\phi\rangle$.

2.4. If we have a two-dimensional Hilbert space and represent states by two-component column vectors such as u and v, find the necessary and sufficient conditions on the 2×2 matrix M so that $u^\dagger M v$ defines an inner product on the Hilbert space.

Hint: Recall the three conditions for the inner product on physical states, but you can just state without proof any properties which follow automatically from matrix multiplication.

2.5. Recall that functions such as the exponential of operators are defined through their Taylor series.

(a) Show that if $\left[\hat{A},\hat{B}\right] = 0$ then

$$\exp\left(\hat{A}\right)\exp\left(\hat{B}\right) = \exp\left(\hat{A}+\hat{B}\right).$$

(b) Show that if $\left[\hat{A},\hat{B}\right] \neq 0$ then

$$\exp\left(\alpha\hat{A}\right)\exp\left(\beta\hat{B}\right) \neq \exp\left(\alpha\hat{A}+\beta\hat{B}\right)$$

for arbitrary $\alpha,\beta \in \mathbb{C}$. (The expressions can be equal for specific values of α,β.)

(c) Show that if $\hat{J}^2 = -\hat{I}$ where $\hat{I}$ is the identity operator then

$$\exp\left(\theta \hat{J}\right) = (\cos\theta)\hat{I} + (\sin\theta)\hat{J}$$

for any $\theta \in \mathbb{C}$.

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3 Problems: Measurement and uncertainty

3.1. Consider a two-dimensional Hilbert space with states represented by two-component column vectors, and with the standard inner product. **PROBLEMS CLASS 1**

(a) Find the (normalised) density matrix for each of the following states:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

(b) Find states corresponding to the following density matrices, if possible:

$$\begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4}i \\ -\frac{\sqrt{3}}{4}i & \frac{1}{4} \end{pmatrix} , \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4}i \\ \frac{\sqrt{3}}{4}i & \frac{1}{4} \end{pmatrix} , \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} , \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} .$$

(c) Calculate $\mathrm{Tr}(\rho^2)$ for each of the density matrices ρ in parts (a) and (b).

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4 Problems: Qubits and the Bloch sphere

4.1. An arbitrary qubit density matrix (for a mixed or pure state) can be written

$$\hat{\rho} = \frac{1}{2}\left(I + \mathbf{r}\cdot\sigma\right)$$

where I is the 2×2 identity matrix, σ_i are the three Pauli sigma-matrices and the real 'Bloch' vector $\mathbf{r}$ has length $|\mathbf{r}| \leq 1$.

(a) Suppose $\hat{\rho}_1$ and $\hat{\rho}_2$ are density matrices for the pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ respectively. What condition on $\hat{\rho}_1\hat{\rho}_2$ is equivalent to the statement that $|\psi_1\rangle$ is orthogonal to $|\psi_2\rangle$?

(b) Express the condition for $\hat{\rho}_1\hat{\rho}_2$ in part (a) (and now allowing pure or mixed states) in terms of conditions on the Bloch vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ defining the two density matrices. How are two orthogonal qubit states represented on the Bloch sphere?

4.2. Consider a qubit system with standard basis $\{|0\rangle, |1\rangle\}$. Define the following states:

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad , \quad |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$
$$|L\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i\,|1\rangle\right) \quad , \quad |R\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i\,|1\rangle\right)$$

(a) Find the density matrices for each of the pure states $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |L\rangle, |R\rangle$.

(b) Find the density matrices for each of the following mixed states:

(i) $|0\rangle$ with probability $\frac{1}{2}$, $|1\rangle$ with probability $\frac{1}{4}$, $|+\rangle$ with probability $\frac{1}{4}$.
(ii) $|0\rangle$ with probability $\frac{1}{2}$, $|1\rangle$ with probability $\frac{1}{2}$.
(iii) $|+\rangle$ with probability $\frac{1}{2}$, $|-\rangle$ with probability $\frac{1}{2}$.
(iv) $|L\rangle$ with probability $\frac{1}{2}$, $|R\rangle$ with probability $\frac{1}{2}$.

(c) Using the Bloch sphere, sketch the regions which can be described as an ensemble (with arbitrary probabilities whose total is 1) of the following pure states:

(i) $|0\rangle$ and $|1\rangle$
(ii) $|+\rangle$ and $|-\rangle$
(iii) $|L\rangle$ and $|R\rangle$
(iv) $|1\rangle$ and $|L\rangle$
(v) $|0\rangle$, $|+\rangle$ and $|R\rangle$

4.3. Suppose a single qubit system is in the state $|0\rangle$.

(a) What are the possible outcomes, and associated probabilities, of a measurement of the observable σ_3?

(b) If instead σ_1 is measured, what are the possible outcomes and probabilities?

(c) Now suppose σ_1 is measured first, then σ_3 is measured and then σ_1 is measured again. Describe the possible outcomes and probabilities at each stage. Is there any relation between the results of the two measurements of σ_1?

5 Problems: Bipartite systems

5.1. For a single classical bit the NOT gate implements the logical operation

PROBLEMS CLASS 3

$$m \to \overline{m} \equiv NOT(m)$$

defined by

$$\overline{0} = 1\ ,\quad \overline{1} = 0.$$

(a) Show that a quantum NOT gate can be implemented as a unitary operation on a single qubit, i.e. sending $|m\rangle \to |\overline{m}\rangle$, and write the unitary operator as a 2×2 matrix in the standard basis.

(b) We can erase a classical bit by sending

$$0 \to 0 \;\text{ and }\; 1 \to 0.$$

Explain why this operation cannot be implemented by a unitary transformation of a single qubit system.

(c) Can we construct a unitary transformation to erase a single qubit in a larger system formed by taking a tensor product of the qubit Hilbert space with another (finite dimensional) Hilbert space?

5.2. Consider a two-qubit bipartite system and use the standard orthonormal basis states for each qubit subsystem.

(a) Explain why any pure state can be written in the form

$$|\Psi\rangle = a\,|0\rangle \otimes |\phi_0\rangle + b\,|1\rangle \otimes |\phi_1\rangle$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are normalised states in system B.

(b) What, if any, constraints must $a, b \in \mathbb{C}$, $|\phi_0\rangle$ and $|\phi_1\rangle$ satisfy so that $|\Psi\rangle$ is normalised.

(c) Write the density operator $\hat{\rho}$ for the system, and calculate the reduced density operators $\hat{\rho}_A$ and $\hat{\rho}_B$.

(d) Show that $\mathrm{Tr}(\hat{\rho}_A^2) = \mathrm{Tr}(\hat{\rho}_B^2)$ and find the range of possible values for this quantity.

(e) What conditions must be satisfied in order to maximise the value of $\mathrm{Tr}(\hat{\rho}_A^2)$ and what property does the state $|\Psi\rangle$ have in this case.

(f) What conditions must be satisfied in order to minimise the value of $\mathrm{Tr}(\hat{\rho}_A^2)$.

(g) Suppose now that system B has a Hilbert space with dimension larger than two, but system A is still an single qubit. Does that change any of the results above?

5.3. Repeat question 5.2 for a bipartite system where system A is a two-qubit system and system B is a two-qubit system or a larger system. Write a suitable generalisation of the form of a pure state $|\Psi\rangle$ in part (a).

6 Problems: Entanglement applications

6.1. Consider the operators **PROBLEMS CLASS 2**

$$\hat{N}_1 \equiv \sigma_1 \otimes \hat{I} \text{ and } \hat{N}_2 \equiv \sigma_3 \otimes \sigma_1$$

acting on a 2-qubit system.

(a) Write $\hat{N}_1$ and $\hat{N}_2$ as 4×4 matrices in the representation where the standard basis states $|m\rangle \otimes |n\rangle$ are written as 4-component column vectors with all components zero except a 1 in the row counted by the 2-digit binary number $(mn)_2$, e.g. $|1\rangle \otimes |0\rangle \to (0\ 0\ 1\ 0)^T$.

(b) Write the operators $\hat{N}_+ \equiv \hat{N}_1 + \hat{N}_2$ and $\hat{N}_\times \equiv \hat{N}_1 \hat{N}_2$ in matrix form. Explain why the structure of the 4×4 matrices shows that $\hat{N}_\times$ can be written in the form $\hat{A} \otimes \hat{B}$ but $\hat{N}_+$ cannot.

(c) Show that $\hat{U} \equiv \frac{1}{\sqrt{2}} \hat{N}_+$ and $\hat{N}_\times$ are unitary operators.

(d) Show that $\hat{U}$ acting on the 4 basis states $|m\rangle \otimes |n\rangle$ produces the 4 Bell states

$$|\beta_{xy}\rangle \equiv \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\overline{y}\rangle \right)$$

and show that none of the Bell states is a separable state.

(e) Find the 4 states produced by $\hat{U}$ acting on the 4 states $|\pm\rangle \otimes |\pm\rangle$ where $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Are any of the resulting states separable?

If any are separable, can you explain why? (Hint: look at $\hat{N}_1$ and $\hat{N}_2$ acting on these states.)

6.2. For a 2-qubit system we define the 4 Bell states in terms of the standard basis states as: **PROBLEMS CLASS 3**

$$|\beta_{xy}\rangle \equiv \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |y\rangle + (-1)^x |1\rangle \otimes |\overline{y}\rangle \right).$$

(a) Write the Bell states in the basis $\{ |{++}\rangle, |{+-}\rangle, |{-+}\rangle, |{--}\rangle \}$.

(b) Write the Bell states in the basis $\{ |LL\rangle, |LR\rangle, |RL\rangle, |RR\rangle \}$.

(c) If Alice and Bob share the Bell state $|\beta_{00}\rangle$ (one qubit each) what are the possible outcomes and the associated probabilities, and what are Alice and Bob's final states in each case:

(i) Alice measures σ_3.

(ii) Bob measures σ_3.

(iii) Alice measures σ_1.

(iv) Alice measures σ_2.

(v) Alice measures σ_1 and then Bob measures σ_3.

(vi) Alice measures σ_1 after Bob measures σ_3.

(vii) Alice measures σ_1 and then Bob measures σ_1.

6.3. Charlie knows that Alice and Bob are meeting for lunch on Saturday. He wants to send them a message with a surprise announcement that they can read together at that time, and not before. The problem is, he will be travelling and unable to communicate with them that day. However, he will see Alice on Thursday and Bob on Friday before he leaves, so he decides to give them each half the message with instructions to combine the information over lunch on Saturday. Suppose Charlie needs 100 bits for his message.

(a) At first Charlie decides to give Alice half the bits, say the odd ones (i.e. the first, the third, the fifth, etc.) and to give Bob the even ones. However, he soon realises that this won't work since Alice and Bob will probably be able to guess some of the message from their 50 bits. After some thought he realises that he can avoid this problem by giving them each 100 bits. How might this work?

(b) Just before Alice arrives on Thursday, Charlie realises the flaw in his plan. Alice and Bob are both impatient so will just send each other the information before Saturday! Then (what luck!) he remembers he has 50 Bell states. He checks his 100-bit message and performs some unitary transformations of the Bell states. When Alice leaves he gives her 50 qubits and some instructions, and the next day gives the remaining 50 qubits to Bob.

Assuming Alice and Bob cannot meet before Saturday, and they do not have a quantum communication channel, explain how Charlie's plan might work.

6.4. Alice needs to urgently send a secret message to Bob. For simplicity, assume she can do this with just 2 bits. They would like to do this using superdense coding, but they don't share any entangled states and to make matters worse someone ordered the wrong equipment – Alice's quantum entangler is broken, as is her quantum receiver and Bob's quantum transmitter. Fortunately, they each share a Bell state with Charlie whom they trust. So, their plan is for Alice to send the message to Charlie using superdense coding, and for Charlie to read it and transmit it to Bob, again using superdense coding. Then, they get news that Eve is spying on Charlie (it is vital that she does not see the message) and also blocking all his quantum communications. All seems lost, but actually there is a way for Alice to send the message to Bob using superdense coding with Charlie's help. Explain how this can be done in a way that does not require Charlie to use quantum communication, and so that Eve gains no information about the message from eavesdropping on any classical communications, or watching what Charlie does.

7 Problems: Information theory

7.1. (a) Show that the operator $\hat{S} = \mathbf{n} \cdot \sigma$, where $\mathbf{n}$ is a three-dimensional unit vector, has eigenvalues ± 1. PROBLEMS CLASS 4

(b) Recall that any qubit density matrix can be written in terms of a Bloch vector $\mathbf{r}$. Calculate the expectation value of $\hat{S}$ in terms of $\mathbf{n}$ and $\mathbf{r}$.

(c) By considering the range of values possible in part (b), state what the eigenstates of $\hat{S}$ are in terms of a relation between $\mathbf{r}$ and $\mathbf{n}$.

(d) For each Bell state, how is the expectation value of $\hat{S} \otimes \hat{I}$ related to that of $\hat{I} \otimes \hat{S}$?

(e) For each Bell state, calculate the expectation value of $\hat{S} \otimes \sigma_3$ and $\hat{S} \otimes \sigma_1$.

7.2. Suppose a device performs a unitary transformation on the tensor product of any input state with a fixed state $|\Omega\rangle$. For two different input states $|\psi\rangle$ and $|\phi\rangle$ the device seems to be a quantum copier, i.e. it maps PROBLEMS CLASS 4

$$|\psi\rangle \otimes |\Omega\rangle \to |\psi\rangle \otimes |\psi\rangle \quad \text{and} \quad |\phi\rangle \otimes |\Omega\rangle \to |\phi\rangle \otimes |\phi\rangle\,.$$

Show that this is only possible if the two states $|\psi\rangle$ and $|\phi\rangle$ are either the same or orthogonal.

7.3. Show that the Shannon entropy of a random variable which can take N different values is maximised if and only if the probability distribution is uniform. Calculate the maximum Shannon entropy (for a given fixed N).

7.4. Suppose we have messages encoded as strings of N bits, where each bit can be considered a random variable having value 1 with probability p.

(a) Calculate the Shannon entropy of a single bit.

(b) What is the Shannon entropy of a string of N bits? (Recall that if X and Y are independent random variables, $H(X, Y) = H(X) + H(Y)$.)

(c) Suppose $N = 1000$ and $p = 3/4$. What is the minimum average length of message which could contain the same information? (You do not have to give a specific method to encode the message.)

7.5. Suppose two bits are described by random variables X and Y.

(a) Calculate the joint Shannon entropy, the conditional entropy of X given Y, the conditional entropy of Y given X, and the mutual information in each of the following cases, where (X, Y) can take (with equal probability) only the values:

(i) $(0, 0), (0, 1), (1, 0), (1, 1)$

(ii) $(0, 1), (1, 0), (1, 1)$

(iii) $(0, 1), (1, 0)$

(iv) $(0, 1), (1, 1)$

(b) Calculate the relative entropy for the above probability distributions of

- Case (iii) to case (ii)
- Case (iii) to case (i)
- Case (ii) to case (i) PROBLEMS CLASS 4

- Case (iv) to case (ii)
- Case (iv) to case (i)

7.6. Consider a qubit with density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1+z & 0 \\ 0 & 1-z \end{pmatrix}$$

(a) Calculate the von Neumann entropy $S(\rho)$ as a function of z.

(b) Show that $S(\rho)$ is a monotonic function for $z \in [0, 1]$ and find the minimum and maximum values of the entropy.

(c) Calculate the entanglement entropy, as a function of θ, of the state

$$|\Psi\rangle = \cos\theta\,|0\rangle \otimes |0\rangle + \sin\theta\,|1\rangle \otimes |1\rangle\,.$$

7.7. Calculate the von Neumann entropy of

(i) $\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

(ii) The ensemble of $|0\rangle$ and $|1\rangle$ with equal probabilities.

and calculate the relative entropy of state (i) to state (ii).

7.8. Consider the state

$$|\Psi\rangle = \cos\theta\,|0\rangle \otimes |0\rangle + \sin\theta\,|1\rangle \otimes |1\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B.$$

Calculate the relative entropy of this state to the state with density matrix

$$\hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B$$

where $\hat{\rho}_A$ and $\hat{\rho}_B$ are the reduced density matrices in each system.