FAKE LENS SPACES AND NON-NEGATIVE SECTIONAL CURVATURE

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ABSTRACT. In this short note we observe the existence of free, isometric actions of finite cyclic groups on a family of 2-connected 7-manifolds with non-negative sectional curvature. This yields many new examples including fake, and possible exotic, lens spaces.

INTRODUCTION

Riemannian manfolds with positive or non-negative sectional curvature have been of great interest to geometers but there are not many examples nor many obstructions known. For non-negatively curved manifolds the most far-reaching structure theorem is that of Gromov bounding the total Betti number by a constant depending only on the dimension. Beyond that, given the dearth of examples and theorems, it is of interest to generate new methods and new examples of non-negatively curved manifolds.

In the recent paper [GKS1] we showed that there is a six parameter family $\mathscr{F}$ of 2-connected 7-manifolds each with the cohomology of an $\mathbf{S}^3$-bundle over $\mathbf{S}^4$ and admitting non-negative sectional curvature. The family $\mathscr{F}$ is a rich source of interesting new examples; it includes all homotopy 7-spheres (each with infinitely many $\mathrm{SO}(3)$-invariant metrics) as well as infinitely many examples with non-standard linking form. The latter class represent the first known examples of 2-connected 7-manifolds with non-negative curvature that are not even homotopy equivalent to $\mathbf{S}^3$-bundles over $\mathbf{S}^4$; see [GKS2]. The manifolds $M^7_{\underline{a},\underline{b}} \in \mathscr{F}$ are each the total space of a Seifert fibration over an orbifold $\mathbf{S}^4$ with generic fiber $\mathbf{S}^3$. The parameters $\underline{a} = (a_1, a_2, a_3), \underline{b} = (b_1, b_2, b_3)$ are each triples of integers satisfying $a_i, b_i \equiv 1 \pmod 4$ for all $i \in \{1, 2, 3\}$, and $\gcd(a_1, a_2 \pm a_3) = 1, \gcd(b_1, b_2 \pm b_3) = 1$. Note that the subfamily corresponding to $a_1 = b_1 = 1$ is precisely the one introduced by K. Grove and W. Ziller in [GZ], and consists of all $\mathbf{S}^3$-bundles over $\mathbf{S}^4$.

1991 *Mathematics Subject Classification.* 53C20, 53C30, 57S15, 57S25.

Key words and phrases. Codimension one biquotient foliations, non-negative curvature, fundamental groups.

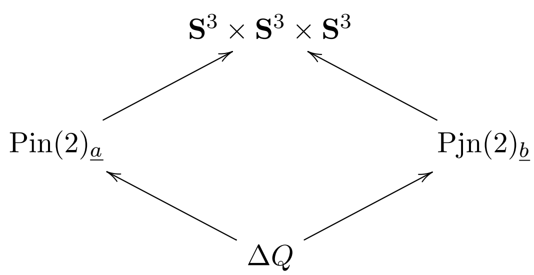
Main Theorem. *There exists a free, isometric action of* $\mathbf{Z}_\ell$ *on* $M^7_{\underline{a},\underline{b}} \in \mathscr{F}$ *if and only if* $\gcd(\ell, a_2 \pm a_3) = 1$ *and* $\gcd(\ell, b_2 \pm b_3) = 1$ *(which implies that* ℓ *is necessarily odd). In particular, there are infinitely many fake lens spaces in dimension* 7 *admitting non-negative sectional curvature (see Table 1).*

Recall that a fake lens space is a manifold with finite, cyclic fundamental group and universal cover a homotopy sphere, while an exotic lens space is a fake lens space that is homeomorphic, but not diffeomorphic, to a lens space. It would be interesting to obtain a classification of these quotients up to diffeomorphism.

Acknowledgments. It is a pleasure to thank the MATRIX institute in Creswick, Australia, and the organizers of the Australian-German workshop "Differential Geometry in the Large" held there in February 2019, where this work was initiated and discussed. The institute and the workshop provided ideal working conditions for collaborating on this project. S. Goette and M. Kerin received support under the DFG Priority Program 2026 *Geometry at Infinity*. K. Shankar received support from the National Science Foundation.[1]

1. $\mathbf{Z}_\ell$ ACTIONS ON THE FAMILY $\mathscr{F}$

Consider the family of 10-manifolds $P^{10}_{\underline{a},\underline{b}}$ with a cohomogeneity-one action of $\mathbf{S}^3 \times \mathbf{S}^3 \times \mathbf{S}^3$ given by the group diagram



where ΔQ is the diagonal embedding of the group $Q = \{\pm 1, \pm i, \pm j, \pm k\} \subseteq \mathbf{S}^3$ and

$$\mathrm{Pin}(2)_{\underline{a}} := \{(e^{ia_1\theta}, e^{ia_2\theta}, e^{ia_3\theta})\} \cup \{(e^{ia_1\theta}, e^{ia_2\theta}, e^{ia_3\theta}) \cdot j\}$$
$$\mathrm{Pjn}(2)_{\underline{b}} := \{(e^{jb_1\theta}, e^{jb_2\theta}, e^{jb_3\theta})\} \cup \{i \cdot (e^{jb_1\theta}, e^{jb_2\theta}, e^{jb_3\theta})\}$$

When $\gcd(a_1, a_2 \pm a_3) = 1$ and $\gcd(b_1, b_2 \pm b_3) = 1$ the subgroup $1 \times \Delta\mathbf{S}^3 \subseteq \mathbf{S}^3 \times (\mathbf{S}^3 \times \mathbf{S}^3)$ acts freely and isometrically with quotient $M^7_{\underline{a},\underline{b}}$. The family $\mathscr{F}$ consists of all such spaces and each $M^7_{\underline{a},\underline{b}} \in \mathscr{F}$ inherits a codimension-one singular Riemannian foliation by biquotients (or double coset manifolds) with regular leaf diffeomorphic

[1] The views expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

to the hypersurface $B_0 := (\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/\Delta Q$, and singular leaves of codimension two diffeomorphic to $B_- := (\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/\mathrm{Pin}(2)_{\underline{a}}$ and $B_+ := (\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/\mathrm{Pjn}(2)_{\underline{b}}$ respectively (both have the integral cohomology of $\mathbf{S}^3 \times \mathbf{RP}^2$). In each case, the free action of the group $U \in \{\Delta Q, \mathrm{Pin}(2)_{\underline{a}}, \mathrm{Pjn}(2)_{\underline{b}}\}$ on $\mathbf{S}^3 \times \mathbf{S}^3$ is described by

$$U \times (\mathbf{S}^3 \times \mathbf{S}^3) \to \mathbf{S}^3 \times \mathbf{S}^3$$
$$\left((u_1, u_2, u_3), \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right) \mapsto \begin{pmatrix} q_1\, u_1^{-1} \\ u_2\, q_2\, u_3^{-1} \end{pmatrix}.$$

There is an obvious isometric action by $\mathbf{S}^3$ on the left of the first factor of $\mathbf{S}^3 \times \mathbf{S}^3$ which commutes with each U action and induces an isometric (leaf-preserving) action of $\mathrm{SO}(3)$ on each $M^7_{\underline{a},\underline{b}} \in \mathscr{F}$. In particular, $-1 \in \mathbf{S}^3$ always acts trivially on $M^7_{\underline{a},\underline{b}}$.

By the Slice Theorem, the quotient $M^7_{\underline{a},\underline{b}}$ is the union of disk bundles over the two singular leaves glued along their common boundary, a regular leaf. Furthermore, as noted in the introduction, each $M^7_{\underline{a},\underline{b}}$ is a cohomology $\mathbf{S}^3$-bundle over $\mathbf{S}^4$. In particular, $M^7_{\underline{a},\underline{b}}$ is a 2-connected 7-manifold with $H^4(M^7_{\underline{a},\underline{b}}; \mathbf{Z}) = \mathbf{Z}_n$, where $n = \frac{1}{8}\det\begin{pmatrix} a_1^2 & b_1^2 \\ a_2^2 - a_3^2 & b_2^2 - b_3^2 \end{pmatrix}$. Whenever $n \neq 0$, $M^7_{\underline{a},\underline{b}}$ is a rational homology sphere, while $n = \pm 1$ ensures that $M^7_{\underline{a},\underline{b}}$ is a homotopy 7-sphere. In particular, all exotic 7-spheres show up in the family $\mathscr{F}$ [GKS1].

The Main Theorem is an immediate consequence of the following observation.

Theorem 1.1. $\mathbf{Z}_\ell \subseteq \mathrm{SO}(3)$ *acts freely and isometrically on* $M^7_{\underline{a},\underline{b}}$ *if and only if* $\gcd(\ell, a_2 \pm a_3) = 1$ *and* $\gcd(\ell, b_2 \pm b_3) = 1$. *In particular,* ℓ *is necessarily odd.*

Proof. Consider the isometric (leaf-preserving) action of $\mathbf{Z}_r = \{w \in \mathrm{U}(1) \subseteq \mathbf{C} : w^r = 1\} \subseteq \mathbf{S}^3$ (the r-th roots of unity) on $M^7_{\underline{a},\underline{b}}$ described on each biquotient leaf $(\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/U$, $U \in \{\Delta Q, \mathrm{Pin}(2)_{\underline{a}}, \mathrm{Pjn}(2)_{\underline{b}}\}$, by

$$\mathbf{Z}_r \times (\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/U \to (\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/U$$
$$\left(w, \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) \mapsto \begin{bmatrix} wq_1 \\ q_2 \end{bmatrix}.$$

As every $\mathbf{Z}_\ell \subseteq \mathrm{SO}(3)$ is, up to conjugation, covered by such a $\mathbf{Z}_r \subseteq \mathbf{S}^3$, the statement of the theorem is now equivalent to the claim that $\mathbf{Z}_r$ acts freely (resp. effectively freely) on $M^7_{\underline{a},\underline{b}}$ if and only if $r = \ell$ (resp. $r = 2\ell$) for some $\ell \in \mathbf{Z}$ satisfying $\gcd(\ell, a_2 \pm a_3) = 1$ and $\gcd(\ell, b_2 \pm b_3) = 1$. As $a_2 \pm a_3$ and $b_2 \pm b_3$ are all even, it is clear that ℓ must be odd.

Since ΔQ is a subgroup of $\mathrm{Pin}(2)_{\underline{a}}$ and $\mathrm{Pjn}(2)_{\underline{b}}$, it suffices to check freeness of the action on the singular leaves, that is, for $U \in \{\mathrm{Pin}(2)_{\underline{a}}, \mathrm{Pjn}(2)_{\underline{b}}\}$. Moreover, the argument for freeness of the action on $B_- = (\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/\mathrm{Pin}(2)_{\underline{a}}$ is completely analogous to the argument for $B_+ = (\mathbf{S}^3 \times \mathbf{S}^3)/\!\!/\mathrm{Pjn}(2)_{\underline{b}}$ by viewing elements of $\mathbf{S}^3$ as being of the form $u + iv$, for $u, v \in \mathrm{span}\{1, j\}$, rather than the usual $u + vj$, for $u, v \in \mathbf{C}$). For this reason, freeness of the action will only be verified on B_-.

To this end, suppose that there is point in B_- with isotropy, that is, that

$$\begin{bmatrix} wq_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

for some $w \in \mathbf{Z}_r$. Then there exists some $z \in \mathrm{U}(1) \subseteq \mathbf{C}$ and some $\lambda \in \{0, 1\}$ such that

$$wq_1 = q_1 j^{-\lambda} \bar{z}^{a_1} \quad \text{and} \quad q_2 = z^{a_2} j^{\lambda} q_2 j^{-\lambda} \bar{z}^{a_3}. \tag{1.1}$$

The goal is now to show that the only possibility at every such point is $w = 1$ (resp. $w \in \{\pm 1\}$) precisely when $\gcd(r, a_2 \pm a_3) = 1$ (resp. $\gcd(r, a_2 \pm a_3) = 2$). Clearly, the analogous conditions involving the parameters $\underline{b}$ arise when considering the singular leaf B_+.

If $\lambda = 0$, then (1.1) implies that $wq_1 = q_1 \bar{z}^{a_1}$ and $q_2 = z^{a_2} q_2 \bar{z}^{a_3}$. Writing $q_2 = u_2 + v_2 j$, with $u_2, v_2 \in \mathbf{C}$, and using the commutation relations for the quaternions yields

$$q_2 = z^{a_2} q_2 \bar{z}^{a_3} \iff u_2 + v_2 j = z^{a_2 - a_3} u_2 + z^{a_2 + a_3} v_2 j.$$

Since $|q_2| = 1$, this forces either $z^{a_2 - a_3} = 1$ or $z^{a_2 + a_3} = 1$.

On the other hand, $wq_1 = q_1 \bar{z}^{a_1} \iff w = q_1 \bar{z}^{a_1} \bar{q}_1$. Therefore,

$$1 = w^r = (q_1 \bar{z}^{a_1} \bar{q}_1)^r = q_1 \bar{z}^{r a_1} \bar{q}_1.$$

Conjugating both sides by $\bar{q}_1$, it follows that $\bar{z}^{r a_1} = 1$.

Setting $d_\pm = \gcd(r a_1, a_2 \pm a_3)$, these identities together yield $z^{d_\pm} = 1$. Moreover, since $\gcd(a_1, a_2 \pm a_3) = 1$, it follows that $d_\pm = \gcd(r, a_2 \pm a_3)$. In particular, it is clear that $d_\pm = 1$ implies $z = 1$ and, hence, $w = 1$, since $w = q_1 \bar{z}^{a_1} \bar{q}_1$. Similarly, if $d_\pm = 2$, then $z \in \{\pm 1\}$ and, hence, $w \in \{\pm 1\}$.

If, on the other hand, $d_\pm > 2$, then the $\mathbf{Z}_r$ action cannot even be effectively free. Indeed, choose $z \in \mathrm{U}(1)$ such that $z^{d_\pm} = 1$ and $z \neq \pm 1$. Since r is divisible by $d_\pm$ by definition, it follows that $z^r = (z^{d_\pm})^{r/d_\pm} = 1$ and, therefore, $z, z^{a_1} \in \mathbf{Z}_r$. Notice, however, that $\gcd(a_1, a_2 \pm a_3) = 1$ implies $\gcd(a_1, d_\pm) = 1$ and, thus, $\gcd(2a_1, d_\pm) = \gcd(2, d_\pm)$. This ensures that $z^{a_1} \neq \pm 1$ since, otherwise, the identities $z^{2a_1} = 1$ and $z^{d_\pm} = 1$ would imply that $z^{\gcd(2, d_\pm)} = 1$ and, hence, that $z \in \{\pm 1\}$, a contradiction.

Now, setting $w = z^{a_1} \in \mathbf{Z}_r \backslash \{\pm 1\}$ yields, for example,

$$w \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} z^{a_1} \\ 1 \end{bmatrix} = \begin{bmatrix} z^{a_1} \bar{z}^{a_1} \\ \bar{z}^{a_2 - a_3} \end{bmatrix} = \begin{bmatrix} 1 \\ \bar{z}^{a_2 - a_3} \end{bmatrix} = \begin{bmatrix} 1 \\ (\bar{z}^{d_\pm})^{(a_2 - a_3)/d_\pm} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Suppose, finally, that $\lambda = 1$. In this case, (1.1) yields the equalities $wq_1 = q_1 \bar{j} \bar{z}^{a_1}$ and $q_2 = z^{a_2} j q_2 \bar{j} \bar{z}^{a_3}$. The first equality can be rewritten as $w = -q_1 z^{a_1} j \bar{q}_1$, which implies that $z^{a_1} j = -\bar{q}_1 w q_1$. Notice that $z^{a_1} j \in \mathrm{span}\{j, k\}$, which implies that $\mathrm{Re}(w) = \mathrm{Re}(\bar{q}_1 w q_1) = 0$ and, therefore, that $w \in \{\pm i\} \cap \mathbf{Z}_r$. However, $\pm i \in \mathbf{Z}_r$ if and only if $r \equiv 0 \bmod 4$, which is impossible if $\gcd(r, a_2 \pm a_3), \gcd(r, b_2 \pm b_3) \in \{1, 2\}$. $\square$

Example 1.2. Actions on homotopy 7-spheres: To illustrate some specific examples we exhibit parameters along with values of ℓ for which there are fake lens spaces with non-negative sectional curvature. Note that in the case when the universal cover is *not* the standard 7-sphere, the manifold cannot be diffeomorphic to a lens space. Therefore, computing differential invariants for these manifolds could yield infinitely many examples of exotic lens spaces with non-negative sectional curvature. We exhibit a few examples of actions on some homotopy spheres (which are determined up to oriented diffeomorphism by the Eells–Kuiper invariant μ, [EK]). Note that the last example is a non-Milnor sphere i.e., a homotopy 7-sphere that is *not* diffeomorphic to an $\mathbf{S}^3$-bundle over $\mathbf{S}^4$.

$(\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$	$(\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$	$\mu(\underline{\mathbf{a}}, \underline{\mathbf{b}})$	free, iometric $\mathbf{Z}_\ell$ action
$(5, -3, 1)$	$(-7, 5, -3)$	$27/28$	all odd ℓ
$(985, -3, 1)$	$(1393, 5, -3)$	$6/28$	all odd ℓ
$(29, -3, 1)$	$(41, 5, -3)$	$20/28$	all odd ℓ
$(17, -47, 33)$	$(-15, -219, 217)$	$5/28$	$\gcd(\ell, 2 \cdot 5 \cdot 7 \cdot 109) = 1$

TABLE 1. Free, isometric $\mathbf{Z}_\ell$ actions on homotopy 7-spheres

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